Characterization of Admissible Marking Sets in Petri Nets with Uncontrollable Transitions

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I. PROOF OF THEOREM 2

This section provides a formal proof of Theorem 2 in the TAC paper. To this aim we need a series of preliminary results that are reported in the following.

Property 1: For any \( Q \subseteq \mathcal{M} \) and any \( t \in T_a \), it holds: \( Q_t \subseteq Q \) and \( Q = \{ m \in Q \mid R(N, m) \subseteq Q \} \).

Property 2: For any \( Q, L \subseteq \mathcal{M} \) and any \( t \in T_u \), it holds: \( A \subseteq Q_t \subseteq Q \) if \( A \subseteq Q \).

Now, according to the definitions in the TAC paper, given a Petri net system \((N, m_0)\) with a legal marking set \( L \), \( \Gamma(L, t) \) denotes the set of legal markings that can lead to forbidden markings by firing \( t \) only. Moreover, \( L_t = L - \Gamma(L, t) \) denotes the subset of legal markings from which no forbidden marking can be reached by firing \( t \) only. Therefore, extending the above notation to a sequence of transitions, \( L_{t_1 \ldots t_n} = (\ldots ((L_{t_2} - \Gamma(L_{t_2}, t_2)) - \Gamma(L_{t_2}, t_2)) \ldots) \), denotes the subset of legal markings from which no forbidden marking can be reached by firing the sequence of transitions \( t_2 \ldots t_n \). The following results hold.

Lemma 1: Given a set of legal markings \( L \), for any set of uncontrollable transitions \( \alpha, \beta, \ldots, \gamma \), it holds: \( L_{\alpha \beta \ldots \gamma} \subseteq L_{\alpha} \subseteq L_{\alpha \beta} \subseteq \ldots \subseteq L_{\alpha \beta \ldots \gamma} = A \)

Proof: It trivially follows from Properties 1 and 2.

The relationship among the marking sets in Lemma 1 is shown in Fig. 1.

Fig. 1. Relationship among \( L, L_{\alpha}, L_{\alpha \beta}, \ldots, L_{\alpha \beta \ldots \gamma} \) and \( A \)

Lemma 2: Given a legal marking set \( L \) and a series of uncontrollable transitions \( \alpha, \beta, \ldots, \gamma \), if \( \forall t \in T_u, \Gamma(L_{\alpha \beta \ldots \gamma}, t) = \emptyset \), then \( L_{\alpha \beta \ldots \gamma} = A \).

Proof: According to Lemma 1, \( A \subseteq \bigcup_{t \in T_u} \Gamma(L_{\alpha \beta \ldots \gamma}, t) = \emptyset \). Therefore, it is \( A \subseteq \bigcup_{t \in T_u} \Gamma(L_{\alpha \beta \ldots \gamma}, t) = \emptyset \).

Theorem 2 [TAC paper]: Given an ordinary Petri net system \((N, m_0)\) and a legal marking set \( L \) as the inputs, under Assumption A1 the output of Algorithm 2 is equal to the admissible marking set of \( L \).

Proof: First of all we observe that Algorithm 2 ends in a finite number of steps due to Assumption 1. Indeed, thanks to such an assumption the set of sub-markings involved in the constraint is finite, i.e., set \( L^* \) is finite. This clearly implies that the first execution of Step 2 (when \( Q = L \)) requires the examination of a finite number of markings when computing \( Q_t \). This leads to the definition of a new set \( Q \) that is equal to \( Q_t \).

Now, the cardinality of the new set \( Q^* \) is in general different with respect to the cardinality of the previous set \( Q^* \), and may eventually be larger. However, once we reach a situation where
the number of restricted places cannot increase anymore, then
the cardinality of $Q'$ always decreases, thus proving that the
algorithm terminates in a finite number of executions.
Finally, the correctness of Step 5 follows from Lemma 2. ■

II. PROOF OF THEOREM 3

Before providing a formal proof of Theorem 3 in the TAC paper
we need some preliminary results, in particular two properties and
one lemma.

Property 3: Let $(\omega', k) = \rho(\omega, k, t, p)$ where $p \in *n^*$. We have
$\sigma'(t) = 0$ if $\sigma(t) > 0$.

Property 4: Let $(\omega', k) = \rho(\omega, k, t, p)$ where $p \in *t \cap *$. We have
$\sigma'(t) = \sigma(t)$ if $\sigma(t) > 0$.

Lemma 3: Given a linear constraint $(\omega, k)$ and an
uncontrollable transition $t$ with $\sigma(t) > 0$, we have $\forall m \in Q_{(\omega', k)},$
$R(N, m) \subseteq Q_{(\omega', k)}$, where $(\omega', k) \in \mathcal{Q}(\omega, k, t)$.

Proof: We have the following two cases:
1) $(\omega', k) = \rho((\omega, k), t, p)$, where $p \in *t$

We have $\sigma'(t) = 0$ by the above Property 3. Hence, $\forall m \in Q_{(\omega', k)},$
$R(N, m) \subseteq Q_{(\omega', k)}$.

2) $(\omega', k) = \rho((\omega, k), t, p)$, where $p \in *t \cap *$

Since $\omega'(p) = k + 1$ by Definition 6 in the TAC paper, it is easy
to see that $\forall m \in Q_{(\omega', k)}, m(p) = 0$. Since $p \in *t$, $t$ cannot fire.
Clearly, $\forall m \in Q_{(\omega', k)}, R(N, m) \subseteq Q_{(\omega', k)}$.

Therefore, the conclusion holds. ■

Theorem 3 [TAC paper]: For any $t \in T_u$ and any $(\omega, k)$
in $\Omega$, it holds: $(\mathcal{Q}_{(\omega, k)}) = \mathcal{Q}_{(W)}$, where $W = \mathcal{Q}(\omega, k, t)$.

Proof: We have two cases: $\sigma(t) \leq 0$ and $\sigma(t) > 0$.
1) $\sigma(t) \leq 0$

Straightforward from Property 2 in the TAC paper, we have
$(\mathcal{Q}_{(\omega, k)}) = \mathcal{Q}_{(\omega, k)}$.

2) $\sigma(t) > 0$

First, we prove that $(\mathcal{Q}_{(\omega, k)}) \supseteq \mathcal{Q}_{(W)}$. $\forall (\omega', k) \in W$, we have
$\omega' \geq \omega$ according to Definitions 6 and 7 in the TAC paper. It can
be inferred that $\forall m, \omega' \cdot m \leq k$, we have $\omega \cdot m \leq k$. That is to say, an
any marking that satisfies some linear constraint in $W$ must satisfy
$(\omega, k)$, i.e., $Q_{(\omega, k)} \supseteq \mathcal{Q}_{(W)}$ holds. According to the
above Lemma 3, it is clear that $\forall m \in \mathcal{Q}_{(W)}, R(N, m) \subseteq \mathcal{Q}_{(W)}$.
Hence, we have $(\mathcal{Q}_{(\omega, k)}) \supseteq \mathcal{Q}_{(W)}$.

Next, we prove that $(\mathcal{Q}_{(\omega, k)}) \subseteq \mathcal{Q}_{(W)}$. By contradiction,
suppose that there exists a marking $m \in (\mathcal{Q}_{(\omega, k)})$ satisfying
$m \in \mathcal{Q}_{(W)}$, i.e.,
$$\forall (\omega', k) \in W, \omega \cdot m \geq k. \quad (1)$$

Let $\alpha$ be a sequence of transitions. It consists of only $t$ and can fire from $m$. $|\alpha|$ is finite since otherwise $m \in (\mathcal{Q}_{(\omega, k)})$.
Moreover, it is clear that there exists $\alpha$ such that $m(\alpha)m$, and $\exists p$
eq t, m(p) = 0. Since $m \in (\mathcal{Q}_{(\omega, k)})$, we have $\omega \cdot m \leq k$. According to Definitions 6 and 7 in the TAC paper, there exists $(\omega_z, k) \in W$
such that $\omega_z \cdot m = \omega \cdot m$. Since $\omega \cdot m \leq k$, we have $\omega_z \cdot m \leq k$. Since
$\omega_z \cdot m = \omega \cdot m + \sigma(\alpha)[t]$ and $\sigma(t) \geq 0$ due to the above Properties
3 and 4, we have $\omega_z \cdot m \geq \omega \cdot m$. Since $\omega \cdot m \leq k$, we have $\omega_z \cdot m \leq k$, which contradicts (1). Hence, $(\mathcal{Q}_{(\omega, k)}) \subseteq \mathcal{Q}_{(W)}$. Therefore, $(\mathcal{Q}_{(\omega, k)}) = \mathcal{Q}_{(W)}$. ■

III. PROOF OF THEOREM 5

Theorem 5 [TAC paper]: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ be a
marking set for $(N, m_0)$ and $t \in T_u$, $Q_i = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ if $\sigma_i(t)$
$\leq 0$ and $\sigma_i(t) > 0$.

Proof: Let $Q_i = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ and $t$ be the inputs of
Algorithm 1. We have $B_1 = (Q_{(\omega_1, k_1)})$ and $B_2 = (Q_{(\omega_2, k_2)})$, as
stated in Steps 1 and 2. Since $\sigma_i(t) \leq 0$ and $\sigma_i(t) > 0$, $B_1 = Q_{(\omega_1, k_1)}$
and $B_2 = Q_{(\omega_2, k_2)}$. Clearly, $C_1 = \emptyset$ and $C_2 = \emptyset$. As a result, the
output $Q_{out} = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$.
According to Theorem 1 in the
TAC paper, we have $Q_{out} = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$. ■