On the robust stabilization of nonlinear uncertain systems with incomplete state availability

The control problem of a class of nonlinear uncertain processes with incomplete state availability is dealt with by means of second order sliding mode control technique. With the aim of using a continuous control, a combined scheme is proposed, in which a 2-sliding mode differentiator and a 2-sliding mode controller are coupled. The stability of the whole observer-controller system is proved for a class of nonlinear plants, and some simulation results are finally provided.

1 Introduction

The control problem of dynamical systems in the presence of uncertainties is one of the most common problems to deal with when considering real plants. In fact, a mathematical model of a real system provides only an approximate representation of actual phenomena. Therefore, the real plant behavior is affected by uncertainties, whose effect on the system performance should be carefully taken into account. For this reason, in recent years the control of uncertain processes has gained great interest in the research community [1-6]. Among them, the sliding mode control (SMC) methodology turns out to be characterized by high simplicity and robustness [5-9]. In its standard formulation, SMC implies discontinuous control actions, switching at very high (theoretically-infinite) frequency between values with opposite sign and with possible high magnitude. This regime leads to undesired high-frequency oscillations of the system output (chattering behavior), and cannot be physically implemented in mechanical systems controlled by forces or torques. Some control schemes devoted to avoid the above-cited drawback have been presented in the literature, among them those based on sliding observers [6,10], smoothing methods [5], and second order SMC (2-SMC) [7,9]. In the first case, the discontinuous control signal is confined in the observer dynamics, and therefore it doesn’t affect the controlled plant directly [6,10]. The second approach, probably the most intuitive, consists in eliminating the discontinuity of the control across the sliding manifold by means of smooth approximations (possibly with time-varying parameters) of the sign function [5]. In the third approach, the discontinuous control signal is the time derivative of the actual control input, which results in being continuous [7,9,11].

All the above methods require the state to be completely available, and if this is not the case, the problem becomes a challenging one. Nonlinear state observers can be used to estimate the unavailable state, but they are difficult to implement when poor knowledge on the system dynamics is available. Moreover, the separation principle cannot be generally invoked in the nonlinear setting, and
the stability of the overall system (observer-controller-plant) is generally hard to guarantee [12,13]. Recently the separation principle was demonstrated for a given class of nonlinear system in case a “sufficiently fast” high gain observer is used for estimating the output derivatives, and a globally bounded state-feedback control is implemented [14].

The objective of this paper is to achieve robust regulation of the system state by continuous feedback, in the presence of hard system uncertainties and with some state variables not measurable.

Consider the single-input nonlinear system

\[ \dot{x} = f(x,t,u) \]  

where \( x=[x_1 \ x_2 \ \ldots \ x_n] \) is the state vector, \( u(t) \in \mathbb{R} \) is the control and \( f(x,t,u) \) is an uncertain smooth vector field.

Let the state vector be completely measurable, and choose a sliding manifold \( \sigma=\sigma(x,t)=0 \) such that the system motion constrained on such a manifold (which is referred in the literature as the “zero dynamics”)

\[ \begin{align*}
\dot{x} &= f(x,t,u(x,t)) \\
\sigma(x,t) &= 0
\end{align*} \]  

is globally asymptotically stable (GAS). Once the desired manifold is properly designed according to the above stability requirements, the goal is to force the \( \sigma \) variable to vanish in finite time by continuous state feedback.

It is easy to check that \( \dot{\sigma} \) and \( \dot{\hat{\sigma}} \) can be expressed as

\[ \dot{\sigma}(t) = \frac{\partial \sigma}{\partial x}(x,t) f(x,t,u) \]  

(3)

\[ \dot{\hat{\sigma}}(t) = \varphi(x,t,u) + \frac{\partial \sigma}{\partial u}(x,t,u) \hat{u}(t) \]  

(4)

If the time derivative of the plant control, \( \hat{u}(t) \), is considered as the control variable [10,15], the 2-SMC approach allows to design a suitable discontinuous control signal \( \hat{u}(t) \) capable of steering both the sliding variable and its unavailable time derivative \( \sigma \) to zero, so that the actual plant control \( u(t) \) is continuous and chattering is avoided (anti-chattering procedure) [7,9,11]. If some state vector components are assumed not measurable, the above direct approach fails, and usually the presence of system uncertainties makes it very difficult to implement a state observer.

In this paper our attention is focused on the possibility of using sliding differentiators for state-feedback stabilization of nonlinear uncertain systems. Recently, special sliding mode observers performing robust differentiation of continuous smooth signals have been presented, either based on first order SMC (1-SMC) [16] or on 2-SMC [17-19]. In this paper a sliding differentiator and a sliding controller are combined, in order to deal with the regulation problem for a class of systems in the presence of uncertainty and incomplete state availability.

Given that the controlled plant has some particular structural properties, that is the unavailable state vector components can be expressed in terms of the time derivative of available quantities, an estimate of the state can be obtained by means of robust real-time differentiation, and the anti-chattering procedure in [11] can be applied considering the approximate sliding manifold

\[ \hat{\sigma} = \sigma(\hat{x},t) \]  

(5)

where \( \hat{x} \) is the estimated state vector. The following Section 2 is devoted to the full statement of the problem.
The observer and controller structures are recalled in Section 3, while in Section 4 the analysis of the overall observer-controller scheme is dealt with. Section 5 and 6 present some simulation results and discussions.

2 Problem formulation

Consider system (1), and assume that any solution of (1) is well-defined for all $t$, provided $u(t)$ is bounded and continuous. Moreover, assume that a component of $x(t)$, say $x_i(t), 1 \leq i \leq n,$ is not available for measurements. In order to use a differentiator as a state observer, it was evidenced in the previous section that the controlled plant has to exhibit the following property: the unmeasurable state can be expressed by the time derivative of available quantities.

A class of plants for which both the choice of the sliding manifold and the differentiator design turn out to be simplified is that of dynamical systems which can be represented in the so-called “canonical Brunovsky form”, i.e.

$$f(x,u) = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x,t) + g(x,t)u(t) \end{bmatrix}$$

(6)

with $x_n$ not available and where $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. A manifold with stable zero dynamics is given by

$$\sigma(x) = x_n + \sum_{i=1}^{n-1} c_i x_i = 0$$

(7)

with $c_i, i = 1, 2, \ldots, n-1$, real positive constants such that

$$P(z) = z^{n-1} + \sum_{i=1}^{n-1} c_i z^{i-1}$$

(8)

is a Hurwitz polynomial. Once on $\sigma(x)=0$, the system behaves like a reduced-order stable linear system and the control objective turns out to be asymptotically achieved.

Consider the first time derivative of $\sigma(x)$, namely

$$\dot{\sigma}(x,t) = f(x,t) + g(x,t)u(t) + \sum_{i=1}^{n-1} c_i x_i \dot{x}_i(t)$$

(9)

Set $y_1 = \sigma$ and $y_2 = \dot{\sigma}$, then the system dynamics (1), (6) along with the second order sliding variable dynamics can be rewritten in the form

$$\begin{cases}
\dot{\bar{x}}(t) = A\bar{x}(t) + b y_1(t) \\
x_n(t) = -c\bar{x}(t) + y_1(t)
\end{cases}$$

(10)

$$\begin{cases}
y_1(t) = y_2(t) \\
y_2(t) = F_1(x,t) + F_2(x,t)u(t) + \frac{\partial g}{\partial x_n}(x,t)\dot{x}_n(t) + g(x,t)\dot{u}(t)
\end{cases}$$

(11)

where $\bar{x} = [x_1, x_2, \ldots, x_{n-1}]$ is the measurable part of the state $c = [c_1, c_2, \ldots, c_{n-1}], A$ is an $(n-1)\times(n-1)$ matrix in companion form with the last row coinciding with vector $-c$, $b=[0,0,\ldots,1]^T \in \mathbb{R}^{n-1}$ and

$$F_1(x,t) = \frac{\partial f}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} x_i + \left( c_{n-1} + \frac{\partial f}{\partial x_n} \right) f(x,t) + \sum_{i=1}^{n-1} c_i x_i$$

(12)

$$F_2(x,t) = \frac{\partial g}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial g}{\partial x_i} x_i + \frac{\partial g}{\partial x_n} f + \left( c_{n-1} + \frac{\partial f}{\partial x_n} \right) g(x,t)$$

(13)

Note that (10) is a stable linear system driven by $y_1(t)$, while (11) corresponds to a nonlinear uncertain second order system with control $u(t)$. Moreover, observe that the two systems are coupled through the state-dependent uncertainty terms. The control task is achieved if $y_1(t)$ is reduced to zero in finite time.

Assume that

$$\frac{\partial g}{\partial x_n} = 0$$

(14)

which is verified in Euler-Lagrange systems, and rewrite
the nonlinear dynamics (11) as
\[
\begin{align*}
\dot{y}_1(t) &= y_2(t) \\
\dot{y}_2(t) &= F_1(x,t) + F_2(x,t)y_2(t) + g(x,t)u(t) = \\
&= \varphi(x,y_2,t) + g(x,t)u(t)
\end{align*}
\tag{15}
\]
where \( F_1(x,t) \) and \( F_2(x,t) \) can be easily derived by (9), (11)-(14).

As to the uncertainties, assume that
\[
\begin{align*}
\|f(x,t)\| &\leq N + M\|s\| \\
0 &< G_1 \leq g(x,t) \leq G_2
\end{align*}
\tag{16}
\]
where \( N, M, G_1, G_2 \) are positive constants, and that
\[
\begin{align*}
\|F_1(x,t)\| &\leq F_{1M}(\|s\|) \\
\|F_2(x,t)\| &\leq F_{2M}(\|s\|)
\end{align*}
\tag{17}
\]
where \( F_{1M}(\|s\|), F_{2M}(\|s\|) \) are known non-decreasing functions, which imply that the drift term \( \varphi(x,y_2,t) \) in (15) is bounded as follows:
\[
\|\varphi(x,y_2,t)\| \leq F_{1M}(\|s\|) + F_{2M}(\|s\|)\|y_2(t)\|
\tag{18}
\]
As the state variable \( x_n(t) \) is not measured, the sliding variable \( \sigma(x) \) is not available. The stabilization problem could be solved if an estimate of \( x_n(t) \) is found, and both the estimation error \( x_n - \hat{x}_n \) and the estimated sliding quantity \( \hat{\sigma} = \sigma(\hat{x}) \) are steered to zero.

3 The Controller and Differentiator Unit

3.1 The Controller Unit

2-SMC of systems of arbitrary order can be reduced to the stabilization problem of a proper second order nonlinear uncertain system like (15). The state variables of this auxiliary system are the sliding variable and its time derivative, and its uncertain dynamics is coupled with that of a reduced-order linear system driven by the sliding variable. Some 2-SMC algorithms with finite time convergence have been presented in the literature [7], provided that an upper bound of the modulus of the drift term \( \varphi(x,y_2,t) \) in (15) can be found such that
\[
|\varphi(t)| \leq \begin{cases} 
F_0(t) & t_0 \leq t < t_{M_1} \\
F_k(t) & t_{M_k} \leq t < t_{M_{k+1}}
\end{cases}
\tag{19}
\]
where \( t_{M_k}, k=1,2,..., \) is the sequence of the time instants at which \( y_2 \) is zero, and \( t_0 \) is the initial control time. In this paper we refer to the “sub-optimal” 2-SMC algorithm [20], which is recalled for the reader’s convenience.

Consider system (1), (6), (14) and assume that the control task is fulfilled by stabilizing the uncertain system (15), (16), (19).

The first step of the 2-SMC approach is that of reaching a known region containing the sliding manifold (initialization phase). In the anti-chattering procedure [11] the control is initialized to ensure that \( \text{sign} \sigma(t_0) = -\text{sign} \sigma(t_0) \), and its time derivative is defined in the successive time interval in order to force the system trajectory to hit in finite time the \( \sigma = 0 \) axis, i.e.
\[
u(t) = \frac{1}{G_1}[F_0(t) + h\text{sign}(\sigma(t)) \quad h > 0 \quad t_0 \leq t < t_{M_1} \tag{20}]
\]
\[
u(t_0) = -\frac{1}{G_1}[\overline{\sigma}(\|s(t_0)\|) + q\text{sign}(\sigma(t_0)) \quad q > 0 \tag{21}]
\]
\[
\overline{\sigma}(\|s(t_0)\|) = N + M\|s(t_0)\| + \sum_{i=1}^{k}G_i\dot{x}_i(t_0)
\tag{22}
\]
From this time instant on, the control derivative is defined according to the “sub-optimal” algorithm
\[
u(t) = -\alpha(t)R_k\text{sign}
\left(\sigma(t) - \frac{1}{2}\sigma_{(t_{M_k})}\right) \quad t_{M_k} \leq t < t_{M_{k+1}} \tag{23}
\]
\[k=1,2,...\]
the finite-time reaching of the sliding set $\sigma = \dot{\sigma} = 0$, which implies the exponential convergence to zero of $\mathbf{x}$, is ensured, provided that the controller parameters are tuned according to the following relationships [11,20]:

$$R_\alpha = \eta \max \left\{ \frac{1}{\alpha G_i}, \frac{4}{3G_i - \alpha G_2} \right\} \Phi_\alpha^T \eta > 1$$  \hspace{1cm} (25)

$$\alpha^* \in \left( 0, \frac{3G_i}{G_2} \right) \cap (0,1]$$  \hspace{1cm} (26)

As a consequence of the fact that an overestimate of $t_{M_{i+1}} - t_{M_i}$ can be simply evaluated [20], it is possible to compute a piecewise-constant upper bound of $|u(t)|$ for the entire control interval

$$|u(t)| \leq U_{\text{in}}(\max(\mathbf{x}(t_0))) \quad t_0 \leq t < t_{M_1}$$  \hspace{1cm} (27)

$$|u(t)| \leq U_{\text{in}}(\max(\mathbf{x}(t_{M_1}))) \quad t_{M_1} \leq t < t_{M_{i+1}}$$

This means that the sub-optimal controller is locally bounded, and this property will be exploited in the following section, in which it is shown that to ensure the stability of the combined observer-controller scheme the availability of such upper bounds is of crucial importance.

### 3.2 The Differentiator Unit

If the 2-sliding set is defined by the vanishing of the error signal and its derivative, the error signal being the difference between the signal to be differentiated and its estimate, a differentiator turns out to be implemented. In this paper we refer to the “super-twisting” 2-sliding differentiator, which is constituted by a couple of integrators driven by a suitable variable structure controller (VSC) (see Fig. 1).

The control aim is to constrain the output $z(t)$ of the first integrator to track the input signal $x(t)$ in finite time. As a consequence, the continuous input $w(t)$ of the first integrator will result in being a $C^0$-estimate of the $x$ derivative.

From the scheme in Fig. 1 it results

$$z(t) = w(t)$$  \hspace{1cm} (28)

Define

$$\begin{align*}
\tilde{e}_1(t) &= z(t) - x(t) \\
\tilde{e}_2(t) &= \tilde{e}_1(t) = w(t) - \dot{x}(t)
\end{align*}$$  \hspace{1cm} (29)

Provided that a positive constant $X_2$ exists such that

$$|\dot{x}(t)| \leq X_2$$  \hspace{1cm} (30)

the differentiator control $w(t)$ can be chosen according to the super-twisting 2-SMC algorithm [19]

$$w(t) = -\lambda |\tilde{e}_1(t)|^{1/2} \text{sign}(\tilde{e}_1(t)) + w_2(t)$$  \hspace{1cm} (31)

$$\dot{w}_2(t) = -W \text{sign}(\tilde{e}_1(t))$$  \hspace{1cm} (32)

$$z(t_0) = x(t_0)$$  \hspace{1cm} (33)

$$w_2(t_0) = 0$$  \hspace{1cm} (34)

The vanishing of both the estimation errors $\tilde{e}_1$ and $\tilde{e}_2$ in finite time is ensured if the differentiator parameters are set in accordance with [19]

$$\begin{cases}
W = \mu X_2 \\
\lambda^2 = \sqrt{\mu} \frac{4X_2(W + X_2)}{W - X_2} \\
\mu > 1
\end{cases}$$  \hspace{1cm} (35)

The reaching time of the perfect derivative estimate can be arbitrarily shortened by increasing the $\mu$ parameter, as it can be shown that
$$t^* \leq \frac{1}{\mu} \left[ t_0 + k(t_1 + X_2) \right]$$

(36)

$k$ being a proper constant [19].

4 The Combined Observer-Controller Scheme

When the stabilization problem of system (1), (6), (14) is faced with partial state availability, due to the system structure we are considering, the use of a differentiator is the procedure of choice to get the on-line estimation of the unmeasurable state variable $x_n$.

The combined scheme we propose consists in coupling the sub-optimal controller and the differentiator based on the super-twisting algorithm (see Fig. 2). To motivate our choice, we need to consider the following facts. Both algorithms require the knowledge of piecewise-constant bounds of the relevant uncertainties, which depend on $x$ and $u$. The computation of such bounds is based on the assumption that the uncertainties are bounded in any bounded domain

$$X \equiv \left\{ x \in \mathbb{R}^n : \|x\| \leq X_m \right\}$$

(37)

$$U \equiv \left\{ u \in \mathbb{R} : \|u\| \leq U_m \right\}$$

(38)

The problem is to find $X_m, U_m$ and, correspondingly, to define the parameters of the control scheme to achieve the twofold task:

1. The evolution of $x$ and $u$ under the action of the proposed algorithm does not leave the pre-specified bounded regions (37), (38).

2. The convergence to the relevant sliding manifold is attained in finite time

To this end, a prediction of the future behavior of the state $x$ and of the control $u$ must be performed. The assumption (16) of linear growth of the nonlinear dynamics is crucial, since the Bellmann-Gronwall Lemma can be used for upper bounding $\|x\|$.

The interaction between observer and controller is evident in the fact that the observer parameters depend on upper bounds of the uncertainties influenced by the control, which, in turn, uses the estimated state given by the observer.

Such logic loop can be solved, for the considered class of system, by choosing the sub-optimal algorithm for control purposes, as $U_m$ can be easily evaluated a-priori, while the use of the super-twisting controller would require the solution of an involved differential equation.

On the contrary, for observation purposes, the super-twisting differentiator is used because it needs no information on the derivative of the available state variable, and, like other 2-SMC schemes, the transient time can be made arbitrarily short by properly increasing the differentiator parameters. An overestimate of the maximum reaching time is available by (36).

The combined observer-controller strategy, according to the following Proposition 1, can be divided into three steps:

1. The plant remains uncontrolled until the differentiator provides a perfect estimate of the unmeasurable $x_n$ at time instant $t^* \leq T^*$, where $T^*$ is the pre-specified maximum transient time.

2. As soon as the perfect estimation of $x_n$ is achieved, the controller is turned on to steer the system into a known neighborhood of the 2-sliding set, and the differentiator parameters are updated correspondingly.
When the first singular point of the sliding variable is reached at time $t_{M_1}$, the control derivative is switched to the sub-optimal 2-SMC until the end of the control interval.

**Proposition 1:** Consider system (1), (6), (14), which verifies assumptions (16)-(17) and with $x_n$ not available. Assume that an upper bound $X_{m_1}$ of the initial value of $x_n$ is known so $|x_n(t_1)| \leq X_{m_1}$. The combined scheme constituted by the super-twisting differentiator (28)-(29), (31)-(34) and the sub-optimal controller (20)-(24) ensures the finite time convergence onto the sliding manifold (7), which implies the asymptotic stability of the system state, provided that:

1. The differentiator input is the $x_{n-1}$ state variable
2. The sub-optimal controller is kept off until a perfect estimate of $x_n$ is available, and is switched on at time $t_0 = T^*$
3. The sliding variable is evaluated by the state estimate $\hat{x} = [x_1, x_2, \ldots, x_{n-1}, w]$
4. The controller parameters are set as

$$\begin{align*}
\bar{T}_0^* (t) &= \bar{T}_{1M} (\|\hat{x}\|) + \bar{T}_{2M} (\|\hat{x}\|) \cdot \tilde{\sigma}_{M}^* \\
\tilde{\sigma}_{M}^* &= N + M \left( \|\hat{x}^*\| + \sum_{i=1}^{n+1} c_i \hat{x}_{i+1}^* \right) + G_2 \phi_i (t^*) \\
\bar{T}_x^* &= \bar{T}_{1M} \left( \|x_M^*\| \right) + \frac{1}{2} a_{h}^{2} + a_{h} \sqrt{4 \bar{T}_{1M} (\|x_M^*\|) + a_{h}^{2}} \\
\mu_{k} &= F_{2M} \left( \|x_{M_k}^*\| \right) + \sqrt{\alpha} G_z \beta^* + 1 \\
\|x_{M_k}^*\| &= Q_{a} \|t_{M_k}^*\| + Q_{a} \tilde{t}_{M_k} \\
\theta_{v} &= Q_{a} \text{ being properly defined constants (see [11])}
\end{align*}$$

5. The differentiator parameters are set as

$$\begin{align*}
W &= \mu \bar{X}_{nM} \\
A^2 &= \sqrt{\mu} \frac{4 \bar{X}_{nM} (W + \bar{X}_{nM})}{W - \bar{X}_{nM}} \\
\mu &= \frac{t_0 + k \left( X_{m} + N + M \left( \|x^*\| + N \left( T^* - t_1 \right) \right) \mu (T^* - t_1) \right)}{T^*} \\
\bar{X}_{nM} &= \left[ N + M \left( \|k(t_1)\| \right) + G_2 \sup_{\gamma \in \Theta} \left( \|u(t)\| \right) \right] + G_2 \sup_{\gamma \in \Theta} \left( \|u(t)\| \right) \ \text{for } t \leq t^* \\
\frac{\sup_{\gamma \in \Theta} \left( \|u(t)\| \right) + G_2 \sup_{\gamma \in \Theta} \left( \|u(t)\| \right) + N} {G_1}
\end{align*}$$

where the function $\Theta (a,b) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as

$$\Theta (a,b) = (a + Nb)^{\mu_{h}}$$

**Proof.** See the Appendix.

**5 Implementation Issues**

The sufficient conditions stated in Proposition 1 are derived on the basis of a worst-case analysis, thus resulting in being very conservative. Computer simulations show that both the controller and differentiator parameters may be set to smaller values than (39)-(48), thus obtaining higher smoothness of the resulting control law and higher accuracy in the practical realization of the control scheme, or even to constant values, reducing the on-line computational effort.

Moreover, some phenomena which are due to the real
nature of the control and measurement devices must be taken into account. More specifically, as to the controller unit, the perfect knowledge of the time instants at which \( \sigma(t) \) is zero would require the availability of an infinite-bandwidth peak detector, which is an ideal device. In real applications, the use of an approximate peak detector must be considered [11,20]. The algorithm for the approximate estimation of the sequence \( t_{M_i} \) is reported below.

**Algorithm 1**

**INITIALIZATION:** \( i=k=0, \sigma(t_{-1})=0 \)

**At every** \( t=i\delta \) **do**

\[
\text{Set } \Delta \sigma_i(t) = \sigma(t_i) - \sigma(t_{i-1})
\]

**If** \( \Delta \sigma_i(t) \Delta \sigma_{i-1}(t) < 0 \) **then**

\[
\begin{align*}
& k_i = k_i + 1 \\
& \hat{t}_{M_i} = t_{i-1}
\end{align*}
\]

\( i := i + 1 \)

where \( \delta > 0 \) is the measurement delay.

It has been proved [11,20] that the estimation error is such that

\[
\left| t_{M_i} - \hat{t}_{M_i} \right| \leq \frac{\delta}{2}
\]

(49)

which implies that

\[
\left| \sigma(t_{M_i}) - \sigma(\hat{t}_{M_i}) \right| \leq O(\delta^2)
\]

(50)

As a result, the real sliding motion occurs in the neighborhood of the sliding set [11,20]

\[
\begin{align*}
& \left| \sigma \right| \leq O(\delta^2) \\
& \left| \sigma \right| \leq O(\delta)
\end{align*}
\]

(51)

which is the typical accuracy featured by real 2-SMC schemes.

The proposed differentiator can be proved to be robust against measurement noises. Assume that the signal to be differentiated is affected by small noise in modulus less than \( \Omega \); the error of the proposed differentiator is directly proportional to \( \sqrt{\Omega} \) whatever the measurement interval [19].

Due to switching imperfections and/or measurement errors, the differentiator unit introduces a bounded error on the estimated derivative, which causes a bounded mismatching between the sliding quantity \( \sigma \) and its estimate

\[
|\sigma - \sigma'| \leq \Sigma
\]

(52)

Algorithm 1 may provide wrong information if the sign of the difference \( \Delta \hat{\sigma}_i(t) \) is affected by such error. Moreover, it is not reasonable to expect higher accuracy in (50) than the noise threshold \( \Sigma \). In fact, in the presence of noise

\[
|\sigma(t_{M_i}) - \sigma(\hat{t}_{M_i})| \leq \max \left[ O(\delta^2) \sigma(\Sigma) \right]
\]

(53)

If the measurement interval in the real implementation of 2-SMC is chosen so that

\[
k\delta^2 = \Sigma
\]

(54)

where \( k \) is a constant which depends on the controller parameters, then the effect of a bounded noise is counteracted [17]. The corresponding real sliding accuracy can be proved to be given by

\[
\begin{align*}
& \left| \sigma \right| \leq O(\Sigma) \\
& \left| \sigma \right| \leq O(\sqrt{\Sigma})
\end{align*}
\]

(55)

Practically, the delay \( \delta \) is heuristically tuned since (54) turns out to be very conservative.
The resulting accuracy is the typical one exhibited by 2-SMC schemes [9], and the above approach appears to be similar to that proposed in [21] for the twisting algorithm.

6 Simulation Results

Consider system (1), (6) with $n=2$ and

$$
\begin{align*}
    f(x,t) &= 3\sin(3t) + x_1 + 2x_2 + 2\cos(x_1) \\
    g(x,t) &= 1 + \frac{1}{2}\sin(x_1)
\end{align*}
$$

(56)

Let the state $x_2$ be unavailable. The initial conditions are set to $x(0)=[1,1]$. The control objective is the asymptotic stabilization of the state vector. To this end, the sliding manifold is chosen as

$$
    \sigma(t) = x_2 + 2x_1
$$

(57)

and it is estimated as

$$
    \hat{\sigma}(t) = w + 2x_1
$$

(58)

where $w$ is the $x_2$ estimate obtained by real-time differentiation of $xT$.

While the sufficient conditions for the convergence stated in Theorem 1 require the on-line adaptation of the controller-observer parameters, a satisfactory behaviour of the system can be attained by using constant values for the relevant parameters, which can be taken much smaller than the highly conservative values predicted by the theoretical analysis. This always occurs when a worst-case design procedure is used.

Constant observer-controller parameters that guarantee the convergence of the proposed scheme have been found to be

$$
\begin{align*}
    R_k &= 130 \\
    \alpha^* &= 1 \\
    W &= 40
\end{align*}
$$

(59, 60, 61)

The practical rule to obtain convergence is to increase $R_k$, $W$, $\lambda$ and $\delta$ and to decrease $\alpha^*$, keeping in mind that the larger the parameters are taken, the worst the accuracy is.

The simulations are carried out by fixed-step Runge-Kutta ODE4 method, with step $d=5 \times 10^{-5}$.

Fig. 3 depicts the actual and estimated sliding variable $\sigma$, with a zoom highlighting the fast transient of the differentiator.

Fig. 4 shows the state behavior, while in Fig. 5 the actual control $u(t)$ and the equivalent control are compared. Note that the actual control turns out to track the equivalent control, which would provide the ideal sliding mode. Due to finite switching frequency of the control derivative, the actual control differs by $O(\delta)$ from the equivalent control.

7 Conclusions

The stabilization problem for a class of nonlinear systems with incomplete state availability has been solved by means of a combined differentiator-controller scheme based on second order sliding modes.

By virtue of the sliding differentiator that behaves almost like a measurement device, and of the chosen controller which allows a suitable on-line adjustment of the differentiator parameters, the closed loop stability of the overall scheme has been demonstrated. The use of robust differentiators lends itself to immediate applications in output feedback control of plants with high relative degree, in which the availability of a certain number of
output derivatives is a crucial point.

References


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APPENDIX

Evaluation of some upperbounds of the state norm

a. Relying on the BIBO nature of the linear subsystem (10), two constants can be found such that the following relationship holds [11]:

...
\[ \|x(t)\| \leq Q_1 \|x(t)\| + Q_2 \sup_{t_j \leq t} \|\sigma(\tau)\| \] (A1)

If the following relationship is satisfied

\[ |\sigma(t)| \leq |\sigma(t_M)| \quad t_{M_i} \leq t \leq t_{M_{i+1}} \] (A2)

then it can be proved that

\[ \|x(t)\| \leq \left(\|x(t)\| + \left( N + G_2 \sup_{t_j \leq t} |u(\tau)| \right) (t - t_j) \right) e^{M(t-t_j)} \] (A4)

**Proof of Proposition 1**

**a. Transient of the differentiator** \( t_i \leq t < T^* \)

The differentiator transient time \( t^* \) is such that

\[ t^* = \frac{1}{\mu} \left[ t_i + k \left( \epsilon_2(t_i) + \sup_{t_j \leq t_{M_i}} \dot{x}_n \right) \right] \] (A5)

\( k \) being a proper constant [19].

By assumption, taking into account (29) and (31)-(34), \( |\dot{\epsilon}_2(t_i)| \leq X_{\dot{\epsilon}_2} \). During this first transient, the control \( u \) is set to zero, and, by (A4)

\[ \|x(t)\| \leq \left( \|x(t)\| + N \left( T^* - t_i \right) \right) e^{M(t-t_i)} \] (A6)

Choose a desired maximum reaching time \( T^* \), and set the \( \mu \) parameter such that

\[ \frac{1}{\mu} \left[ t_i + k \left( X_{\dot{\epsilon}_2} + N + M \left( \|x(t)\| + N \left( T^* - t_i \right) \right) e^{M(t-t_i)} \right) \right] \leq T^* \] (A7)

An upper bound of \( |\dot{x}_n| \) in the whole time interval \( t_i \leq t \leq T^* \) is given by (46), and the differentiator parameters can be chosen accordingly as in (44), (45).

**REMARK** From \( t = T^* \) on, the differentiator provides the exact estimate of \( x_n \), so that \( \dot{x} = x \) and \( \dot{\sigma} = \sigma \), and the control strategy coincides with that in [11], with \( x \) substituted by its estimate \( \dot{x} \). The effectiveness of the controller is ensured if the differentiator parameters are properly updated to take into account the variation of \( \dot{x}_n \) due to the activation of the control \( u \), thus maintaining the condition of perfect estimate.

**b. Reaching the \( \dot{\sigma} = 0 \) axis** \( T^* \leq t < t_{M_i} \)

The idea is to prove that the axis \( \dot{\sigma} = 0 \) is reached in a finite time \( t_{M_i} \). By virtue of (9), taking into account (16), the control \( u(T^*) \) is set to ensure that \( \text{sign} \dot{\sigma}(T^*) = -\text{sign} \dot{\sigma}(t_i) \), and an upper bound of \( |\dot{x}(t)| \) is expressed by (40).

By considering (18), (20)-(22) and (39), (40) \( \dot{\sigma} \) converges to zero in finite time \( t_{M_i} < T^* + \left( \sigma_M^* / h \right) \)

By (A4)

\[ \|x(t)\| \leq \left( \|x(t)\| + N \frac{\sigma_M^*}{h} \right) e^{M \sigma_M^*/h} ; \quad T^* \leq t < t_{M_i} \] (A8)

In the considered time interval, \( |u(t)| \) can be upper bounded as in (47) by considering (9), (16), (A8) and (40), thus \( \dot{x}_n \) can be modified as in (46).

**c. General case-contraction property** \( t_{M_i} \leq t < t_{M_{i+1}} \)

It has been proved that a singular point \( \sigma(t_{M_i}) \), that is a point \( \text{s.t.} \ \dot{\sigma}(t_{M_i}) = 0 \) is reached at the time instant \( t_{M_i} \).

From \( t = t_{M_i} \) on, it will be proved that a sequence of singular values \( \sigma(t_{M_i}) \), \( k = 1, 2, \ldots \) is generated, featuring the contraction condition

\[ |\sigma(t_{M_{i+1}})| \leq \gamma^2 |\sigma(t_{M_i})| ; \quad \gamma^2 < 1 \quad k = 1, 2, \ldots \] (A9)
This implies that the system trajectory approaches the origin of the \( \sigma - \sigma \) plane.

Let \( \Phi_k \) be the maximum modulus of the drift term \( \varphi(t) \) in (15) in the considered time interval \( T_{M_k} = [t_{M_k}, t_{M_{k+1}}] \).

The system trajectory results in being confined between two parabolic limiting curves (see Fig 6).

By considering the worst-case trajectory along the limiting curves, the following relationships hold:

\[
\dot{\sigma}_{M_k} = \sqrt{\sum_{i=1}^{n} |\Phi_k + G_i \alpha' R_k|^2} \tag{A10}
\]

\[
\frac{\sigma(t_{M_{k+1}})}{\sigma(t_{M_k})} \leq \left[ \frac{G_i R_k}{G_i R_k + \Phi_k} + 2 \Phi_k \left( \frac{G_i - \alpha' R_k}{2(G_i R_k - \Phi_k)} \right) \right] \tag{A11}
\]

A sufficient condition for the contraction condition (A9) to occur is the fulfillment of the following system of inequalities:

\[
\begin{align*}
0 &< \alpha' \leq 1 \\
\alpha' G_1 R_k &> \Phi_k \\
\left( \alpha' G_1 R_k^2 - \Phi_k \right) &< 1 \\
\end{align*} \tag{A12}
\]

The above relationships are based on the existence and knowledge of the upper bound \( \Phi_k \). To prove that such a bound exists, and to find an overestimate \( \Phi_k \), an algebraic loop has to be solved. In fact, the control amplitude must depend on the uncertainty bound, but, at the same time, the uncertainty bound depends on the control through the uncertain, state-dependent, term \( \varphi(t) \).

The following chain of inequalities can be written:

\[
\Phi_k \leq F_1(x, t) + \frac{1}{2} \dot{\sigma}_{M_k} \leq F_1(x, t) + \frac{1}{2} \dot{\sigma}_{M_k} \leq F_1(x, t) + \frac{1}{2} \dot{\sigma}_{M_k} \leq \ldots \leq \frac{1}{2} \dot{\sigma}_{M_k} \leq \Phi_k \]

By considering the asymptotic behavior of the latter inequality in (A13) for \( \Phi_k \) tending to infinity, it can be claimed that there exists a positive number \( \Phi_k^* \) such that (A13) is verified for any \( \Phi_k \geq \Phi_k^* \). We choose \( \Phi_k = \Phi_k^* \), and the solution of the second order algebraic equation which derives by

\[
F_k = F_1(x, t) + \frac{1}{2} \dot{\sigma}_{M_k} \leq \Phi_k \leq \Phi_k^* \tag{A14}
\]

is given by (41)-(42).

In this time interval, the state norm can be upper bounded as in (A3). By (9), (16) and (A10), the control satisfies the bound in (47), and \( \dot{X}_{nM} \) can be modified as in (46), (47).

The constant \( \gamma^2 \) in (A9) can be evaluated as

\[
\gamma^2 = \frac{1}{2} \max_{i} \left\{ \max \left\{ \frac{G_i R_k}{G_i R_k - \Phi_k} + 2 \Phi_k \left( \frac{G_i - \alpha' R_k}{2(G_i R_k - \Phi_k)} \right) \right\} \right\} \tag{A15}
\]

so that, recursively,

\[
\begin{align*}
\left| T_{M_{k+1}} \right| &\leq \gamma^{2k} \left| T_{M_k} \right| \\
\left| T_{M_k} \right| &\leq \gamma^{2k-1} \left| T_{M_{k-1}} \right| \\
\end{align*} \tag{A16}
\]

The finite time convergence is a straightforward consequence of (A15)-(A16), which imply that the sum of the time intervals between two subsequent singular values, \( \tau_{M_{k+1}} - \tau_{M_k} \), is upper bounded by a summable geometric series [20].
List of figures

Figure 1
\[ \bar{x} = [x_1, x_2, \ldots, x_{n-1}] \]

Figure 2
Barlini et al., On the robust stabilization of nonlinear uncertain systems with incomplete state availability

Figure 3
\[ x_1(t) \]

\[ x_2(t) \]

Time [sec]

Figure 4
Figure 5
Figure 6