Controlling Chaos Via Second-Order Sliding Modes

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Abstract

The problem of ordering chaos by second-order sliding-mode control is addressed. The proposed approach offers an effective, yet simple, solution tackling both the uncertainties affecting the chaotic dynamics and the measurement noise corrupting the feedback signals. The procedure is illustrated making reference to Chua's circuit and the "suboptimal" second-order sliding-mode control algorithm. Simulations are provided to confirm the good performance and robustness of the proposed scheme.

1. Introduction

The idea of controlling chaos was first proposed in 1990 by Ott, Grebogi and Yorke [1]. They expressed the chaos control problem in terms of stabilising one of the unstable periodic orbits within the strange attractor of the chaotic dynamics, and the task was fulfilled by perturbing one accessible parameter around its nominal value. Various extensions, improvements and applications have been developed following that line of research (see [2] and references therein).

Roughly speaking, the literature reports two main methodologies to control the chaotic dynamics: (i) parameter variation and (ii) state-feedback control. However, the main trend of the literature is to use the latter method rather than make a direct on-line modification to the system parameters.

Most of existing approaches make use of linear control theory, and rely upon a local linearization of the system model around the target trajectory [3]-[5]. However, there are many well-known limitations on applying linearization techniques: they are strictly local, potentially unstable and usually not powerful enough to handle broad classes of complex and/or uncertain dynamics.

Conversely, nonlinear feedback control offers undoubted advantages over linear techniques in terms of precision, robustness and inherent stability properties. In particular, robust control approaches appear to be specially suited to manage uncertain chaotic systems. Adaptive control and Variable Structure Control (VSC) have been extensively applied in this field [6]-[10].

Most of chaos control schemes based on variable structures refer to the conventional relay control, which requires just a rough model of the system dynamics and completely rejects all matched disturbances.

Unless general nonlinear systems, chaotic systems mostly evolve either on a strange attractor or on a periodic orbit, and the state variables are confined to a known bounded region. Thus, constant bounds to the state-dependent uncertainties can be easily found [6], [9], [10]. This sharply reduces the complexity of the resulting Sliding Mode Control (SMC) design. Care must be taken to ensure that the feedback control preserves the stability of the strange attractor.

Second-order Sliding Mode Control (2-SMC) is rather a recent branch of VSC theory [11], [12]. One of the main advantages of the 2-SMC approach is the possibility of counteracting the chattering phenomenon. A further advantage of 2-SMC is that it can deal with first-order actuators with uncertain dynamics. This is particularly helpful in the case of controlled current generators obtained, in practice, by voltage switching techniques.

In this paper, full-state and partial-state feedback controllers based on the unchattering 2-SMC method are proposed either to achieve the robust tracking of a generic desired trajectory (by the full-state feedback realization) or to simply stabilize a periodic trajectory whose actual profile depends on the system dynamics and cannot be arbitrarily set. In many problems involving chaos the latter goal is sufficient, and can be attained by a simplified controller that depends only on a single state variable.

The proposed approach is capable of handling a broad class of chaotic and hyperchaotic systems by taking explicitly into account model uncertainties, noises and external disturbances. Moreover, the direct discretisation of the continuous-time 2-SMC controllers is feasible under mild additional requirements [13].

2. Chaos control: problem formulation

Let us consider the following general autonomous chaotic system
\[
\dot{x}(t) = Ax(t) + f[x(t)]
\]  

where \(x(t) \in \mathbb{R}^n\), \(A\) is a known time-invariant square matrix, and \(f: \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a continuous, possibly uncertain, vector field collecting both the uncertain and nonlinear dynamics of the system.

The problem of chaos control is to force a chaotic system to exhibit a prefixed non-chaotic behavior and to do so robustly versus initial conditions, model uncertainties, parameter variations, and, possibly, measurement errors. Obviously, it is necessary to add some modifiable quantities to the equations defining the chaotic dynamics, so let us consider the "controlled" chaotic system
\[
\dot{x}(t) = Ax(t) + f[x(t)] + Bu(t)
\]
\[
y(t) = Cx(t)
\]

where \(u(t) \in \mathbb{R}^p\) is the control vector, \(y(t) \in \mathbb{R}^m\), \(p \geq n\), is the vector collecting the output variables, and \(B\) and \(C\) are constant matrices of appropriate dimensions. Let \(m \geq 2\), and let \(y_d(t)\) be the target output trajectory. The chaos control problem can be formalized as follows:

**Problem 1:** Given the chaotic system (2) and the desired non-chaotic output trajectory \(y_d(t)\), find a continuous feedback control \(u(t)\) such that
where $T^*$ is a finite transient length and $\varepsilon$ is a positive quantity to be reduced as much as possible.

If $p=n$, the system state is available and the problem becomes a full-state feedback control problem. In a standard output tracking control problem, it is generally assumed that $n>p$ and that the system uncertainty is matching (i.e., $f(x(t))=B\cdot u(t)$). In the considered case, the number and locations of the inputs (i.e., matrix $B$) can be considered as design parameters. Then, the choice $p=m$ reduces Problem 1 to a simplified output tracking control problem with matching uncertainties, and no particular specification about the desired behavior $y_d(t)$ is required. In the literature, chaos control is often addressed choosing as a target output trajectory either an unstable periodic cycle or an unstable equilibrium point of the unforced system (1).

### 2.1 Tracking problem

In this case, the target is an unstable periodic orbit $\tilde{x}_r(t)$. The control objective is to force $x(t) \rightarrow \tilde{x}_r(t)$ as $t \rightarrow T^*$. The tracking problem is first faced assuming that the unstable orbit equations are known a priori. We refer to the following second-order approximation [15]

$$\tilde{x}_r(t) = a_1 \cos(\xi) \cos(\omega t) - b_1 \sin(\xi) \sin(\omega t) + c_1 \sin(\xi) \cos(\omega t) - d_1 \sin(\xi) \sin(\omega t)$$

where $a=2.6$, $b=1.2$, $c=0.6$, $d=0.3$, $\xi=\pi/18$, and $\omega=1.77$. Moreover, in the case of Chua’s circuit, it will be shown that the tracking of a periodic orbit can be achieved by a scalar control and a scalar output.

#### 2.2 Stabilisation problem

When the objective is to find a control $u$ for stabilising the state of the system at one of the unstable equilibrium points, the tracking problem is called stabilisation problem. It can be considered as a particular case of tracking problem with a reference signal $x = const$.

### 3. Output tracking for nonlinear uncertain dynamic systems via 2-SMC

The SMC approach makes use of a discontinuous control action to reduce to zero a suitably-defined output variable (referred to as the “sliding variable”) whose vanishing guarantees the attainment of the tracking objective. The sliding variable usually consists of a linear combination between the tracking error and a certain number of its derivatives.

In classical first-order sliding-mode control (1-SMC), the discontinuous signal acts on the first time-derivative of the sliding variable [16]. Conversely, in 2-SM the discontinuous control affects the second derivative of the sliding variable, and this fact offers the useful possibility of defining the derivative of the actual control signal as an auxiliary discontinuous control, thereby virtually eliminating the chattering effect.

To be more specific, consider system (2) and define the sliding variable as follows

$$s = y - y_d$$

The first and second derivatives of $s$ are expressed by

$$\dot{s} = C \cdot (Ax + f(x)) - \dot{y}_d + C \cdot B \cdot u(t)$$

$$\ddot{s} = C \cdot \frac{\partial f}{\partial x} (Ax + f(x)) + C \cdot B \cdot u(t)$$

Under the assumption that the uncertain drift term in (11) is bounded in any bounded domain, i.e.,

$$C \cdot \left[ Ax + \frac{\partial f}{\partial x}(Ax + f(x)) + B \cdot u(t) \right] \leq \Phi, \quad i=1,2,...,m$$

if matrix $CB$ is positive definite and a known scalar $\mu$ exists such that matrix $CBA + \Phi$ (being the identity matrix) is dominant diagonal then the following multi-input 2-SMC scheme may be used to guarantee the asymptotic vanishing of the vector $s$ by a continuous control $u(t)$ [18]

$$u = -V_s \text{diag}(\alpha(t)_1,...,\alpha(t)_m) \text{sgn}(s - \xi s_a)$$

where $i=1,2,...,m$, $s_{a}$ is the vector of the most recent extremal values (i.e., local maxima, minima and horizontal flex points) of the sliding vector $s$ elements considered as time functions, $\alpha^*$ are defined on the basis of the uncertain matrix $CB$ elements, and $V_s$ is a sufficiently large gain dependent on the uncertainty bounds $\Phi_s$, i.e.,

$$\alpha^* = \{ (s_0(i) - \xi s_a(i)) \}, \quad i=1,2,...,m$$

with $0 < CB_s \leq CB \leq CB_s \Phi$, $(i, j=1,2,...,m)$. In particular, if matrix $CB$ is diagonal, the multi-input tracking Problem 1 can be reduced to $m$ decoupled SISO problems, in which each control law can be designed independently [17].

Making reference to the single-input case, the discrete-time implementation of the “Sub-optimal” 2-SMC (9)-(10) has been considered in previous papers by some of the authors [13], and the direct discretisation of the continuous control schemes has been shown to be feasible under mild additional requirements.

In particular, if $T_s$ is the chosen sampling period, the finite-time convergence to a boundary layer of the second-order sliding manifold $S = \dot{S} = \bar{0}$, whose size is

$$\| \dot{s} \| \ll O(T_s^{-1})$$

where

$$\| s \| \ll O(T)$$

can be achieved. A simple digital algorithm for the approximate evaluation of the extremal values $s_{a}$ of the sliding variable can be found in [13].
3.1 Chaos control in Chua’s circuit

In order to show the applicability of the proposed approach, attention is now focused on Chua’s circuit. The system equations in dimensionless form are [18]

\[
\begin{align*}
\frac{d\tilde{x}_1}{d\tau} &= -\alpha \tilde{x}_1 + x_2 - f(x_1) \\
\frac{d\tilde{x}_2}{d\tau} &= \tilde{x}_1 - x_2 + x_3 \\
\frac{d\tilde{x}_3}{d\tau} &= -\beta \tilde{x}_2 - \gamma \tilde{x}_3,
\end{align*}
\]

where \(f(x_1)\) is expressed as

\[
f(x_1) = m_1 x_1 + \frac{1}{2} \left( m_2 - m_1 \right) \left| x_1 \right| - \left| x_1 - 1 \right| - \left| x_1 + 1 \right|.
\]

If the dimensionless system parameters are set to \(\alpha = 9\), \(\beta = 14\), \(\gamma = 0.1\), \(m_1 = 5/7\), \(m_2 = -8/7\), the system exhibits a chaotic behaviour of the double-scroll type.

3.2 Robust chaos control in Chua’s circuit

In order to show the general applicability of the proposed approach, attention is now focused on its application to chaos control in Chua’s circuit.

3.2.1 Tracking problem

A full tracking of smooth arbitrary trajectories requires the observation of all state variables and a vector control \(u = [u_1, u_2, u_3]^T\). The dynamic system equations become

\[
\begin{align*}
\frac{d\tilde{x}_1}{d\tau} &= \alpha \left[ -x_1 + x_2 - f(x_1) \right] + u_1 \\
\frac{d\tilde{x}_2}{d\tau} &= x_1 - x_2 + x_3 + u_1 \\
\frac{d\tilde{x}_3}{d\tau} &= -\beta \tilde{x}_2 - \gamma \tilde{x}_3 + u_3,
\end{align*}
\]

and \(y(t) = x(t)\), i.e., \(B = C = I_3\) in (2).

The sliding variables are defined as

\[
s_i = x_i - \tilde{x}_i, \quad i = 1, 2, 3,
\]

and the associated second-order dynamics are

\[
\begin{align*}
\frac{d\tilde{s}_1}{d\tau} &= \alpha \tilde{s}_1 + \alpha \tilde{\tilde{s}}_1 + \beta \tilde{s}_2 + \gamma \tilde{s}_3 + \tilde{u}_1 \\
\frac{d\tilde{s}_2}{d\tau} &= \alpha \tilde{s}_1 + \beta \tilde{s}_2 + \gamma \tilde{s}_3 + \tilde{u}_2 \\
\frac{d\tilde{s}_3}{d\tau} &= \alpha \tilde{s}_1 + \beta \tilde{s}_2 + \gamma \tilde{s}_3 + \tilde{u}_3,
\end{align*}
\]

Taking into account that the reduced dynamics of the controlled system turn out to be BIBO stable, and assuming that the desired trajectories are perfectly smooth, it is possible to define known upper bound \(\Phi_i\) (i=1,2,3) to the drift terms in (16). Hence, the use of the controller (7),(11) and (12) assures the finite-time tracking of \(\tilde{x}(t)\).

If the third of (14) is defined in its differential form \(\tilde{x}_3 = -\beta \tilde{x}_2 - \gamma \tilde{x}_3\) and if the parameters \(\beta, \gamma\) are perfectly known, disregarding the initial condition, it is possible to track the desired trajectory by measuring and controlling the first two state variables only.

Furthermore, if the control task is the tracking of any periodic trajectory, a scalar control and a scalar output \(y\) (e.g., \(u = [u_1, 0, 0]^T\) and \(y = x_1\)) are sufficient. Indeed, the controlled system has the form (2) with \(B = [1,0,0]^T\) and \(C = [1,0,0]\), and if the control \(u_1\) guarantees the periodicity of the state component \(x_1\), also the state components \(x_2\) and \(x_3\) will be periodic. In this case, the sliding variable is defined as

\[
s_1 = x_1 - \tilde{x}_1
\]

Its second-order dynamics is

\[
\begin{align*}
\frac{d\tilde{s}_1}{d\tau} &= \alpha \left[ -x_1 + x_2 - f(x_1) \right] + u_1 - \tilde{u}_1 \\
\frac{d\tilde{s}_2}{d\tau} &= x_1 - x_2 + x_3 + u_1 \\
\frac{d\tilde{s}_3}{d\tau} &= -\beta \tilde{x}_2 - \gamma \tilde{x}_3 + u_3 - \tilde{u}_3,
\end{align*}
\]

and the dynamics of the \(x_2\) and \(x_3\) state variables are

\[
\begin{align*}
\frac{d\tilde{s}_2}{d\tau} &= \alpha \tilde{s}_1 + \beta \tilde{s}_2 + \gamma \tilde{s}_3 + \tilde{u}_2 \\
\frac{d\tilde{s}_3}{d\tau} &= \alpha \tilde{s}_1 + \beta \tilde{s}_2 + \gamma \tilde{s}_3 + \tilde{u}_3.
\end{align*}
\]

The system dynamics (18)-(19) are BIBO stable, and the previous considerations apply. Therefore, if \(s_1\) is steered to zero or to a small residual set, the tracking of a periodic orbit is guaranteed.

3.2.2 Stabilisation problem

The stabilisation of a point in the state space can be considered as a special case of the tracking problem in which \(R = const\). Taking into account the previous treatment, it is apparent that any point of the state space can be reached in a finite time by defining the sliding variables as in (15). Conversely, in the particular case of stabilisation on one of the three saddle foci of Chua’s circuit, equations (17)-(19) show that it can be reached asymptotically by controlling only the \(x_1\) state variable.

4. Simulation Results

The discretised controllers were implemented with a sampling period of \(0.3\) s. The initial conditions of the circuit were \(x(0) = [-0.1, -0.1, -0.1]^T\). The constant magnitudes of \(\tilde{u}_1\), \(\tilde{u}_2\) and \(\tilde{u}_3\) were set to 50.

4.1 Tracking problem

The control objective is to find a feedback control \(u = [u_1, u_2, u_3]^T\) such that the trajectory of Chua’s circuit tracks the unstable limit cycle (6). Figure 1 shows the periodic trajectory of the controlled circuit, and Figure 2 shows the \(u_1\) component of the actual control signal. As can be noticed, the signal is periodic and continuous. Figure 3 gives the tracking errors in two cases: with and without noise affecting the measurement of \(x_1(t)\) (a gaussian noise with about 1% of the peak value of the state). The proposed digital control scheme proved to be effective and robust, and does not require any detailed information about the system parameters.

It should be noted that the use of the digital approximate peak-holder [13] might be responsible for the failure of the control scheme if the sampling period is not properly chosen. Using a multi-rate sampling (the peak-holder runs at lower sampling rate) allows us to recover stability.
The single-input partial-state-feedback control scheme with the scalar sliding variable being defined as in (6), (17) has been also implemented. The control signal and the output variable are $u = [u_1 \ 0 \ 0]^T$ and $y = x_1$, respectively. Note that $x_i$ and $\tilde{x}_i$ are periodic but different from the periodic trajectories $\tilde{x}_i$ and $\tilde{x}_i$ in (6). Actually, indeed, $\tilde{x}_2$ and $\tilde{x}_3$ are the solutions of (19) when $x_1 - \tilde{x}_1 \to 0$. Figure 4 shows the state vector components.

5. Conclusions

A second-order sliding-mode approach has been proposed to control uncertain chaotic systems. The higher-order sliding controller does not need a perfect knowledge of the system parameters, but just rough informations about the system structure (relative degree) and upper bounds to the uncertainties. Under these conditions, it allows the tracking of any desired trajectory as well as the stabilisation of any prefixed point. The chaotic Chua circuit has been used as an example to illustrate the applicability and effectiveness of the proposed control approach. Digital simulations confirm the theoretical results.

References