TRACKING CONTROL OF THE UNCERTAIN HEAT AND WAVE EQUATION VIA POWER-FRACTIONAL AND SLIDING-MODE TECHNIQUES

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Abstract. In the present paper, preliminary results towards the generalization to the infinite-dimensional setting of some well-known robust finite-dimensional control algorithms are illustrated. More specifically, we deal with the tracking problem for some classes of linear uncertain infinite-dimensional systems evolving in Hilbert spaces. We design distributed variable-structure stabilizers that are shown to be effective in the presence of external disturbances. The main focus of the present paper is on the rejection of nonvanishing external disturbances. The generalization to the infinite-dimensional setting of the well-known finite-dimensional controllers, namely the “power-fractional” controller [S. P. Bhat and D. S. Bernstein, SIAM J. Control Optim., 38 (2000), pp. 751–766] and two “second-order sliding-mode” control algorithms (the “twisting” and “supertwisting” algorithms [L. Fridman and A. Levant, Higher order sliding modes as a natural phenomenon in control theory, in Robust Control via Variable Structure and Lyapunov Techniques, Springer-Verlag, Berlin, 1996, pp. 107–133; A. Levant, Internat. J. Control, 58 (1993), pp. 1247–1263]), is the main contribution of the present investigation. First, the “distributed twisting” control algorithm is developed to address the asymptotic state tracking of the perturbed wave equation. Next, the finite-time state tracking of the unperturbed heat equation is provided by means of a “distributed power-fractional” controller. Finally, the “distributed supertwisting” controller is suggested to address the asymptotic state tracking of the heat equation in spite of the presence of persistent disturbances. Constructive proofs of stability are developed via the Lyapunov functional technique, which leads to simple tuning rules for the controller parameters. Simulation results are discussed to verify the effectiveness of the proposed schemes.

Key words. infinite-dimensional systems, heat equation, wave equation, uncertain systems, second-order sliding modes

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1. Introduction. Sliding-mode control has long been recognized as a powerful control method to counteract nonvanishing external disturbances and unmodeled dynamics [19]. This method is based on the deliberate introduction of sliding motions into the control system, and, since the motion along the sliding manifold proves to be uncorrupted by matched disturbances, the closed-loop system is guaranteed to exhibit strong properties of robustness against significant classes of disturbances and model uncertainties. Due to these advantages and simplicity of implementation, sliding-mode controllers have widely been used in various applications [21].

On the other hand, many important systems and industrial processes, such as flexible manipulators and chemical reactors, are governed by PDEs and are often de-
scribed by models with a significant degree of uncertainty. Therefore, significant interest has emerged in extending the discontinuous control methods to infinite-dimensional systems. Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented [11, 18, 14, 17], and it is generally shown to retain the main robustness features similar to those possessed by its finite-dimensional counterpart.

Other robust control paradigms have been fruitfully applied in the infinite-dimensional setting, such as in adaptive control [9, 10] and $H_\infty$ and LMI-based design [7, 8]. The recent book [16] presents an overview on existing results in the field of robust control of infinite-dimensional systems.

In the present work we refer to some classes of finite-dimensional robust control algorithms (namely, the Bhat–Bernstein “power-fractional” controller [4] and the “twisting” and “supertwisting” second-order sliding-mode (2-SM) controllers [6, 12]). The mentioned 2-SM controllers are of special interest because in the finite-dimensional setting they significantly improve the performance of SM control systems, in terms of accuracy and chattering avoidance, as compared to the more conventional “first-order” SM control techniques [2, 3, 15].

In the present paper we show that appropriate infinite-dimensional generalizations of the above-mentioned controllers are capable of solving the state-tracking problem for infinite-dimensional systems of practical relevance such as the wave and heat equations subject to a class of nonvanishing perturbations.

A major difficulty has been overcome in that discontinuous control terms of the type $\text{sign}(z(t))$, with $z(t) \in \mathbb{R}^m$, being a customary variable evolving in a finite-dimensional space, do not readily translate in the corresponding term $\text{sign}(z(\xi,t))$ when the involved variable $z$ is defined in an infinite-dimensional Hilbert space $H$. In the present paper, as a general guideline to extend finite-dimensional discontinuous controllers to the infinite-dimensional setting, we suggest replacing any finite-dimensional discontinuous control term $\text{sign}(z(t))$ by the distributed equivalent form $z(\xi,t)/\|z(\xi,t)\|_2$ that can be understood as a unit-vector implementation of the discontinuous control component.

The Lyapunov-based analysis of the control systems in question shows that special restrictions, inherently connected to the infinite-dimensional nature of the controlled dynamics, should be imposed on the admissible external disturbances and reference trajectories. Those sufficient conditions, which do not have a finite-dimensional counterpart, represent an intriguing peculiarity of the proposed analysis and design framework that also incorporates the use of open-loop feedforward injection control terms. Throughout the paper, constructive proofs of stability are developed via the Lyapunov functional technique, which leads to simple tuning rules for the controller parameters. Noticeably, the “distributed supertwisting” controller that will be designed throughout the paper leads to a control input that is a nonsmooth but continuous function of the space and time variables while retaining properties of robustness similar to those featured by discontinuous SM control techniques such as the distributed unit vector control [18, 14].

It should be pointed out that the obtained results assume distributed sensing and actuation, which are both rarely available in practice and make the proposed approaches of mainly theoretical interest. However, the developed methods open the way to further improvements, analogous to similar ones attained in the finite-dimensional setup, such as the boundary control and/or the pointwise output measurement feedback implementation of the proposed feedback controllers, which will be addressed in future research activities.
The rest of the paper is outlined as follows. In section 2 the "distributed twisting" (DTW) control algorithm is developed to address the asymptotic state tracking of the uncertain wave equation. Section 3 shows the relevant simulation results. In section 4, the state-tracking control problem for the heat equation is addressed. In particular, section 4.1 presents the "distributed power-fractional" (DPF) controller and shows that it can guarantee the finite-time tracking of the state reference trajectories for the unperturbed heat equation. Section 4.2 introduces the "distributed supertwisting" (DSTW) controller and shows that it can provide the asymptotic tracking control for the uncertain heat equation in spite of the presence of a class of persistent disturbances. Section 5 discusses the relevant simulation results for the DPF and DSTW controllers. Section 6 concludes the present paper by discussing the main features of the developed schemes and by drawing in a more detailed way the most promising lines of investigation for possible extensions of the obtained results.

1.1. Notation. The notation used throughout is fairly standard. $L_2(a,b)$, with $a \leq b$, stands for the Hilbert space of square integrable functions $z(\zeta)$, $\zeta \in [a,b]$, with the corresponding $L_2$ norm

$$\|z(\cdot)\|_2 = \sqrt{\int_a^b z^2(\zeta) d\zeta}.$$  

$W^{l,2}(a,b)$ stands for the Sobolev space of absolutely continuous scalar functions on $[a,b]$ with square integrable derivatives of the order $l \geq 1$.

2. Perturbed wave equation. A class of uncertain infinite-dimensional systems, defined in a Hilbert space $H = L_2(0,1) \times L_2(0,1)$, governed by a perturbed version of the hyperbolic PDE commonly referred to as the wave equation is considered:

$$y_{tt}(\xi,t) = \nu^2 y_{\xi\xi}(\xi,t) + u(\xi,t) + \psi(\xi,t),$$

where $\xi \in [0,1]$ is the one-dimensional space variable, $t > 0$ is the time variable, and $(y,y_t) \in H$, $t \geq 0$, is the state vector with the norm $\| (y,y_t) \|_H = \| y(\cdot,t) \|_2 + \| y_t(\cdot,t) \|_2$. The coefficient $\nu^2 \in \mathbb{R}$ stands for the elasticity, $u(\xi,t)$ is the modifiable source term (the distributed control input), and $\psi(\xi,t)$ represents a distributed uncertain disturbance source term.

Remark 1. Generally speaking, the disturbance term $\psi$ may depend on the state variables $y$ and $y_t$, too; however, being computed along dynamics $y(\xi,t)$ and $y_t(\xi,t)$ of the state equation (2), it can be viewed as a function of the independent space and time variables $\xi$ and $t$ only.

The initial conditions (ICs)

$$y(\xi,0) = \varphi_0(\xi) \in W^{2,2}(0,1), \quad y_t(\xi,0) = \varphi_1(\xi) \in W^{2,2}(0,1)$$

are assumed to meet the boundary conditions (BCs) imposed on the system. Throughout, we consider two types of BCs, namely, homogeneous Neumann BCs

$$y_\xi(0,t) = y_\xi(1,t) = 0$$

and Dirichlet BCs

$$y(0,t) = y_0(t) \in W^{2,2}(0,\infty), \quad y(1,t) = y_1(t) \in W^{2,2}(0,\infty).$$
In order to deal with strong, sufficiently smooth solutions of the uncertain PDE (2)–(4) (or (5) instead of (4)) in the open loop when no input $u$ is applied, the uncertain term, which appears in the right-hand side of (2), is additionally supposed to satisfy the condition

\[(6) \quad \psi(\xi, t) \in L_2(0, \infty)\]

and as well to meet the BCs imposed on the system.

The control task is to make the scalar field $y(\xi, t)$ follow a given reference $y_r(\xi, t)$ which is required to be smooth enough and such that

\[(7) \quad y_{rt}(0, t) = y_{rt}(1, t) = 0\]

Clearly, the reference $y_r(\xi, t)$ should be selected in accordance with the chosen BCs, namely, Neumann BCs

\[(8) \quad y_r(0, t) = y_0(t), \quad y_r(1, t) = y_1(t).\]

If the control task is achieved, the deviation variable

\[(10) \quad \tilde{y}(\xi, t) = y(\xi, t) - y_r(\xi, t)\]

is eventually driven to zero by the designed controller. The dynamics of the error variable (10) are then written as follows:

\[(11) \quad \tilde{y}_{tt}(\xi, t) = \nu^2 \tilde{y}_{\xi\xi}(\xi, t) + u(\xi, t) + \psi(\xi, t) - y_{rt}(\xi, t) + \nu^2 y_{r\xi}(\xi, t),\]

where, by (8) or (9), the associated BCs are either of Neumann type

\[(12) \quad \tilde{y}_{\xi}(0, t) = \tilde{y}_{\xi}(1, t) = 0\]

or Dirichlet type

\[(13) \quad \tilde{y}(0, t) = \tilde{y}(1, t) = 0.\]

The class of admissible disturbances is specified by the following assumption.

**Assumption 1.** There exists an a priori known constant $M$ such that the following relationship holds uniformly in time:

\[(14) \quad \|\psi(\cdot, t)\|_2 \leq M \quad \forall t \geq 0.\]

If the distributed input function $u(\xi, t)$ is sufficiently smooth, then the above equation locally possesses a unique strong solution $y(\xi, t)$ which can be defined in a standard manner, according to the next definition.

**Definition 1.** A continuously differentiable function $y(\xi, t)$, defined on $[0, T)$ for some $T > 0$, is a strong solution of the boundary value problem (2)–(4) (or (5) instead of (4)) with a continuously differentiable input $u(\xi, t)$ iff it satisfies the equation for $t \in (0, T)$ and, at the same time,

\[(15) \quad \lim_{t \to 0} \|(y(\cdot, t) - \varphi_0(\cdot)), (y_t(\cdot, t) - \varphi_1(\cdot))\|_H = 0.\]
2.1. Robust control of the wave equation. Assume that the state vector \((y, y_t)\) is available for measurements. Then the error state \(\tilde{y}(\xi, t)\) and its time derivative \(\tilde{y}_t(\xi, t)\) are also available for feedback. In order to stabilize the error dynamics the distributed controller

\[
(16) \quad u(y, y_t, \xi, t) = y_{rr\xi}(\xi, t) - \nu^2 y_{\xi\xi}(\xi, t) - \lambda_1 \frac{\tilde{y}(\xi, t)}{\|\tilde{y}(\cdot, t)\|_2} - \lambda_2 \frac{\tilde{y}_t(\xi, t)}{\|\tilde{y}_t(\cdot, t)\|_2}
\]

is proposed, where \(\nu^2\) represents the elasticity coefficient and \(\lambda_1, \lambda_2\) are constant control coefficients. Controller (16) can be viewed as the sum of two components: a distributed feedforward component, and a feedback component consisting of a distributed version of the finite-dimensional twisting 2-SM controller, which is further referred to as the distributed twisting (DTW) controller.

The nonsmooth nature of the DTW controller (16), which undergoes discontinuities on the two manifolds \(\tilde{y} = 0\) and \(\tilde{y}_t = 0\), requires appropriate comments about the meaning of the corresponding solutions for the resulting discontinuous feedback system. The precise meaning of the solutions of (2)–(4) (or (5) instead of (4)) with the piecewise continuously differentiable control input (16) can be defined in the generalized sense of [16] as a limiting result obtained through a certain regularization procedure, similar to that proposed for finite-dimensional systems [5, 19]. According to this procedure, the strong solutions of the boundary value problem are considered only whenever they are beyond the discontinuity manifolds \(\tilde{y} = 0\) and \(\tilde{y}_t = 0\), whereas in a vicinity of these manifolds the original system is replaced by a related system, which takes into account all possible imperfections (e.g., delay, hysteresis, saturation, etc.) in the new input function \(u^\delta(y, y_t, \xi, t)\), for which there exists a strong solution \(y^\delta(\xi, t)\) of the corresponding boundary value problem with the smoothed input \(u^\delta(y, y_t, \xi, t)\). The following definitions are extracted from [16].

**Definition 2.** An absolutely continuous function \(y^\delta(\cdot, t) \in L_2(0, 1)\), defined on some time interval \([0, \tau]\), is said to be an approximate \(\delta\)-solution of (2)–(4), (16) (or (5) instead of (4)) if it is a strong solution of the corresponding boundary value problem with a continuous approximation \(U^\delta(\cdot)\) substituted for the discontinuous components

\[
(17) \quad U(\tilde{y}) = \frac{\tilde{y}(\xi, t)}{\|\tilde{y}(\cdot, t)\|_2}, \quad U(\tilde{y}_t) = \frac{\tilde{y}_t(\xi, t)}{\|\tilde{y}_t(\cdot, t)\|_2}
\]

of the control input (16) such that \(\|U^\delta(\tilde{y}) - U(\tilde{y})\|_2 \leq \delta\) and \(\|U^\delta(\tilde{y}_t) - U(\tilde{y}_t)\|_2 \leq \delta\) for all \(y, y_t \in L_2(0, 1)\) subject to \(\|y\|_2 \geq \delta\) and \(\|y_t\|_2 \geq \delta\), respectively, where \(\delta > 0\).

**Definition 3.** An absolutely continuous function \(y(\cdot, t) \in L_2(0, 1)\), defined on some time interval \([0, \tau]\), is said to be a generalized solution of (2)–(4), (16) (or (5) instead of (4)) if there exists a family of approximate \(\delta\)-solutions \(y^\delta(\cdot, t)\) of the corresponding boundary value problem such that

\[
(18) \quad \lim_{\delta \to 0} \|y^\delta(\cdot, t) - y(\cdot, t)\|_2 = 0, \quad \lim_{\delta \to 0} \|y^\delta_t(\cdot, t) - y_t(\cdot, t)\|_2 = 0
\]

uniformly in \(t \in [0, \tau]\).

In other words, the discontinuous control input (16) is smoothed in an arbitrary manner within the \(\delta\)-vicinities of the discontinuity manifolds \(\tilde{y} = 0\) and \(\tilde{y}_t = 0\), whereas beyond these vicinities it is approximated by a closely related signal. In particular, a relevant approximation occurs when \(U^\delta(\tilde{y}) = \frac{\tilde{y}(\xi, t)}{\delta + \|\tilde{y}(\cdot, t)\|_2}\) and, respectively, \(U^\delta(\tilde{y}_t) = \frac{\tilde{y}_t(\xi, t)}{\delta + \|\tilde{y}_t(\cdot, t)\|_2}\). A generalized solution of the system in question is then obtained through
the limiting procedure by diminishing $\delta$ to zero, thereby making the characteristics of the new system approach those of the original one. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a *sliding mode*.

**Remark 2.** The existence of generalized solutions, thus defined, has been established within the abstract framework of Hilbert space-valued dynamic systems (cf., e.g., [16, Theorem 2.4]), whereas the uniqueness and well-posedness appear to follow from the fact that, in the system in question, no sliding mode occurs except in the origin $y = y_t = 0$. While being well-recognized for 2-SM control algorithms if confined to the finite-dimensional setting, this fact, however, remains beyond the scope of the present investigation.

The asymptotic stability of the generalized solutions of the perturbed wave equation (2)–(4), (10) subject to the control strategy (16) is demonstrated in the next theorem.

**Theorem 1.** Consider the perturbed wave equation (2) along with the ICs and BCs (3) and (4) (or (5)) fulfilling the condition (6), and with the uncertain disturbance satisfying Assumption 1. Consider the reference profile $y_r(\xi,t)$, subject to restrictions (7), and the associated error variable (10). Then, the distributed control strategy (16) with the parameters $\lambda_1$ and $\lambda_2$ such that

$$\lambda_2 > M, \quad \lambda_1 > \lambda_2 + M$$

guarantees the global asymptotic stability of the error dynamics (11).

**Proof.** Let us refer to the error dynamics (11) along with the BCs (12) or (13). Consider the following Lyapunov functional:

$$\tilde{V}(t) = \lambda_1 \sqrt{\int_0^1 \tilde{y}_\xi^2(\xi,t) d\xi + \frac{1}{2} \int_0^1 \tilde{y}_t^2(\xi,t) d\xi + \frac{1}{2} \nu^2 \int_0^1 \tilde{y}_\xi^2(\xi,t) d\xi}.$$  

In order to simplify the notation, the dependence of the system signals from the space and time variables $(\xi, t)$ will be omitted from this point on. Let us rewrite the functional $\tilde{V}(t)$ in compact form as follows:

$$\tilde{V}(t) = \lambda_1 \|\tilde{y}\|_2 + \frac{1}{2} \|\tilde{y}_t\|_2^2 + \frac{1}{2} \nu^2 \|\tilde{y}_\xi\|_2^2.$$  

The time derivative of $\tilde{V}$ is given by

$$\dot{\tilde{V}}(t) = \frac{\lambda_1}{\|\tilde{y}\|_2} \int_0^1 \tilde{y} \tilde{y}_t d\xi + \int_0^1 \tilde{y}_t \tilde{y}_\xi d\xi + \nu^2 \int_0^1 \tilde{y}_\xi \tilde{y}_\xi d\xi.$$  

By evaluating the time derivative (22) along the solutions of the error system (11), (16), it turns out that

$$\dot{\tilde{V}}(t) = -\lambda_2 \|\tilde{y}_t\|_2 + \int_0^1 \tilde{y}_t \psi d\xi + \nu^2 \int_0^1 \tilde{y}_\xi \tilde{y}_\xi d\xi + \nu^2 \int_0^1 \tilde{y}_\xi \tilde{y}_\xi d\xi.$$  

The last term in the right-hand side of (23) can be integrated by parts, and taking into account the homogeneous BCs (12) (or (13)) it yields

$$\nu^2 \int_0^1 \tilde{y}_\xi \tilde{y}_\xi d\xi = \nu^2 (\tilde{y}_\xi(1,t)\tilde{y}_t(1,t) - \tilde{y}_\xi(0,t)\tilde{y}_t(0,t)) - \nu^2 \int_0^1 \tilde{y}_\xi \tilde{y}_\xi d\xi$$

$$= -\nu^2 \int_0^1 \tilde{y}_t \tilde{y}_\xi d\xi.$$
which leads to the simplified form for $\dot{V}(t)$:

$$
(25) \quad \dot{V}(t) = -\lambda_2 \|\tilde{y}_t\|_2 + \int_0^1 \tilde{y}_t \psi d\xi.
$$

From the Hölder integral inequality [1] and taking into account (14), it yields

$$
(26) \quad \left| \int_0^1 \tilde{y}_t \psi d\xi \right| \leq \int_0^1 |\tilde{y}_t \psi| d\xi \leq M \|\tilde{y}_t\|_2.
$$

Then, by virtue of (25) and (26), it follows that

$$
(27) \quad \dot{V}(t) \leq -(\lambda_2 - M) \|\tilde{y}_t\|_2,
$$

which implies that the error dynamics are stable and the Lyapunov functional $V(t)$ is a nonincreasing function of time along the error dynamics, i.e.,

$$
(28) \quad \dot{V}(t_2) \leq \dot{V}(t_1) \quad \forall t_2 \geq t_1 \geq 0.
$$

Denote

$$
(29) \quad D_R = \left\{ (\tilde{y}, \tilde{y}_t) \in H : \dot{V}(\tilde{y}, \tilde{y}_t) \leq R \right\}.
$$

By virtue of (28), it is clear that once an arbitrary $R \geq V(0)$ is fixed the resulting domain $D_R$ proves to be invariant for the error system trajectories. Thus, the next analysis will take into account that the state $(\tilde{y}, \tilde{y}_t)$ remains in the initial domain $D_R$ forever.

We now demonstrate a simple lemma to be used throughout the proof.

**Lemma 1.** If the states $(\tilde{y}, \tilde{y}_t)$ belong to the domain $D_R$ (29), then the following condition holds:

$$
(30) \quad \int_0^1 \tilde{y}_t \tilde{y}_t \ d\xi \geq -\frac{1}{2} \left[ \frac{R}{\lambda_1} \|\tilde{y}\|_2 + \|\tilde{y}_t\|_2^2 \right].
$$

**Proof.**

$$
(31) \quad \dot{V}(t) = \lambda_1 \|\tilde{y}\|_2 + \frac{1}{2} \|\tilde{y}_t\|_2^2 + \frac{1}{2} \nu^2 \|\tilde{y}_t\|_2^2 \leq R \quad \Rightarrow \quad \lambda_1 \|\tilde{y}\|_2 \leq R \quad \Rightarrow \quad \|\tilde{y}\|_2 \leq R \frac{\lambda_1}{R}.
$$

By applying the well-known inequality $ab \geq \frac{1}{2}(a^2 + b^2)$ it follows that

$$
(32) \quad \int_0^1 \tilde{y}_t \tilde{y}_t \ d\xi \geq -\frac{1}{2} \left[ \|\tilde{y}\|_2^2 + \|\tilde{y}_t\|_2^2 \right] = \frac{1}{2} \left[ \|\tilde{y}\|_2^2 \|\tilde{y}_t\|_2^2 + \|\tilde{y}_t\|_2^2 \right].
$$

Being coupled together, (31) and (32) immediately result in (30), which proves the lemma.

Since the state $(\tilde{y}, \tilde{y}_t)$ belongs to the domain $D_R$ (29), the condition

$$
(33) \quad \|\tilde{y}_t\|_2^2 \leq \sqrt{2R} \|\tilde{y}_t\|_2
$$

holds true. Condition (33) is reproduced from the trivial chain of implications:

$$
\dot{V}(t) = \lambda_1 \|\tilde{y}\|_2 + \frac{1}{2} \|\tilde{y}_t\|_2^2 + \frac{1}{2} \nu^2 \|\tilde{y}_t\|_2^2 \leq R \quad \Rightarrow \quad \frac{1}{2} \|\tilde{y}_t\|_2^2 \leq R
$$

$$
\Rightarrow \quad \|\tilde{y}_t\|_2 \leq \sqrt{2R} \Rightarrow \quad \|\tilde{y}_t\|_2^2 \leq \sqrt{2R} \|\tilde{y}_t\|_2.
$$
Now consider the “augmented” functional

\[ V_R(t) = \tilde{V}(t) + \kappa_R \int_0^t \tilde{y} \tilde{y}_t \, d\xi = \lambda_1 \| \tilde{y} \|_2 + \frac{1}{2} \| \tilde{y}_t \|_2^2 + \frac{1}{2} \nu^2 \| \tilde{y}_\xi \|_2^2 + \kappa_R \int_0^t \tilde{y} \tilde{y}_t \, d\xi, \]

where \( \kappa_R \) is a positive constant. In light of Lemma 1, function \( V_R(t) \) can be estimated as

\[ V_R(t) \geq \lambda_1 \| \tilde{y} \|_2 + \frac{1}{2} \| \tilde{y}_t \|_2^2 + \frac{1}{2} \nu^2 \| \tilde{y}_\xi \|_2^2 - \frac{\kappa_R}{2} \left[ \frac{R}{\lambda_1} \| \tilde{y} \|_2 + \| \tilde{y}_t \|_2 \right] \]

\[ = \left( \lambda_1 - \frac{\kappa_R R}{2 \lambda_1} \right) \| \tilde{y} \|_2 + \frac{1}{2} (1 - \kappa_R) \| \tilde{y}_t \|_2^2 + \frac{1}{2} \nu^2 \| \tilde{y}_\xi \|_2^2. \]  

Provided that coefficient \( \kappa_R \) is selected sufficiently small according to

\[ \kappa_R \leq \min \left\{ \frac{2 \lambda_1^2}{R}, 1 \right\}, \]

the augmented functional (35) is then positive definite within the invariant domain \( D_R \), and it turns out to be radially unbounded as \( R \to \infty \). Functional (35) can thus be used as a radially unbounded Lyapunov functional to analyze the global asymptotic stability of the error dynamics. Let us compute the time derivative of \( V_R(t) \) along the trajectories of the error system (11), (12) (or (13) instead of (12)), (16):

\[ \dot{V}_R(t) = -\lambda_2 \| \tilde{y}_t \|_2 + \int_0^1 \tilde{y}_t \tilde{y}_\psi \, d\xi + \kappa_R \int_0^1 \tilde{y}_t^2 \, d\xi + \kappa_R \int_0^1 \tilde{y} \tilde{y}_tt \, d\xi \]

\[ = -\lambda_2 \| \tilde{y}_t \|_2 + \kappa_R \| \tilde{y}_t \|_2^2 + \kappa_R \int_0^1 \tilde{y} \left[ \nu^2 \tilde{y}_\xi + u + \psi \right] \, d\xi \]

\[ = -\lambda_2 \| \tilde{y}_t \|_2 + \kappa_R \| \tilde{y}_t \|_2^2 + \kappa_R \nu^2 \int_0^1 \tilde{y} \tilde{y}_\xi \, d\xi - \kappa_R \lambda_1 \| \tilde{y} \|_2 \]

\[ - \frac{\kappa_R \lambda_2}{\| \tilde{y}_t \|_2} \int_0^1 \tilde{y} \tilde{y}_t \, d\xi + \kappa_R \int_0^1 \tilde{y} \tilde{y}_\psi \, d\xi. \]  

After that, let us compute upper bounds to the sign-indefinite terms of (38). Equation (26) was previously derived, which can be rewritten in a similar form with the signal \( \tilde{y} \) replacing \( \tilde{y}_t \):

\[ \kappa_R \int_0^1 \tilde{y} \tilde{y}_\psi \, d\xi \leq \kappa_R \int_0^1 \tilde{y} \tilde{y}_t \, d\xi \leq \kappa_R \lambda_1 \| \tilde{y} \|_2. \]

\[ \frac{\kappa_R \lambda_2}{\| \tilde{y}_t \|_2} \int_0^1 \tilde{y} \tilde{y}_t \, d\xi \leq \frac{\kappa_R \lambda_2}{\| \tilde{y}_t \|_2} \int_0^1 \| \tilde{y}_t \|_2^2 \, d\xi \leq \frac{\kappa_R \lambda_2}{\| \tilde{y}_t \|_2} \sqrt{\int_0^1 \tilde{y}_t^2 \, d\xi} \sqrt{\int_0^1 \tilde{y}_t^2 \, d\xi} = \kappa_R \lambda_2 \| \tilde{y} \|_2. \]  

\[ \dot{V}_R(t) \leq - \left( \lambda_2 - M - \kappa_R \sqrt{2 R - \kappa_R M} \right) \| \tilde{y}_t \|_2 - \kappa_R \nu^2 \| \tilde{y}_\xi \|_2^2 - \kappa_R (\lambda_1 - \lambda_2 - M) \| \tilde{y} \|_2. \]  

(41)
Therefore, employing the parameter tuning condition (19) and introducing one more restriction

\[
\kappa_R \leq \min \left\{ \frac{2\lambda_1^2}{R}, 1, \frac{\lambda_2 - M}{M + \sqrt{2}R} \right\}
\]

about the coefficient \(\kappa_R\) beyond (37), it readily follows from (41) that the time derivative of the Lyapunov functional (35), computed along the error dynamics, is negative definite. Since the parameter \(R\) can be chosen arbitrarily large, whereas the Lyapunov functional (35) becomes radially unbounded as \(R \to \infty\), the global asymptotic stability of the error dynamics, governed by (11) along with the BCs (12) or (13), is thus concluded. This completes the proof of Theorem 1.

3. Simulation results—wave equation. Consider the perturbed wave equation (2) with homogeneous Neumann-type BCs as in (12). The elasticity parameter is set to \(\nu^2 = 1\). A constant set-point has been considered in the form \(y_r(\xi, t) = y^*_r = 20\).

The ICs are set as

\[
y(\xi, 0) = \cos(2\pi\xi), \quad y_t(\xi, 0) = 0.
\]

The PDE has been solved by means of the standard finite-difference approximation technique, the spatial solution domain \(\xi \in [0, 1]\) being discretized into 40 “solution nodes.” The resulting 40th-order system has been solved by using the fixed-step Euler integration method with a temporal integration step of 0.01s.

In TEST 1, the perturbation \(\psi(\xi, t)\) is set to zero. According to (19), the DTW control strategy (16) is implemented using the parameter values \(\lambda_1 = 10, \lambda_2 = 5\). Figure 1(left) reports the attained solution that converges to the set-point after about 20 seconds.

In TEST 2, a space- and time-varying perturbation is considered in the form

\[
\psi(\xi, t) = 100\sin(2\pi\xi)\sin(2\pi t).
\]

Its \(L^2\) norm bound \(M\), which is involved in the controller tuning inequality (19), is given by \(M = \frac{100}{\sqrt{2}}\). Then the controller gains can be set according to (19) as

\[
\lambda_1 = 400, \quad \lambda_2 = 200.
\]

Figure 1(right) reports the attained solution in TEST 2, which supports the robustness features of the proposed synthesis.

4. Perturbed heat equation. Consider the space- and time-varying scalar field \(Q(\xi, t)\) evolving in a Hilbert space \(H = L_2(0, 1)\), where \(\xi \in [0, 1]\) is the one-dimensional space variable and \(t > 0\) is time. Let it be governed by the following parabolic PDE, which is commonly referred to as the “heat equation”:

\[
Q_t(\xi, t) = \theta_1 Q_{\xi\xi}(\xi, t) + u(\xi, t) + \psi(\xi, t),
\]

where \(\theta_1\) is a positive coefficient called thermal conductivity (or, more generally, diffusivity), \(u(\xi, t)\) is the modifiable source term (the distributed control input), and \(\psi(\xi, t)\) represents a distributed uncertain disturbance source term. The IC

\[
Q(\xi, 0) = \omega_0(\xi) \in W^{2,2}(0, 1)
\]
is assumed to meet the BCs imposed on the system. We consider two types of BCs, namely, homogeneous Neumann BCs

\[ Q_\xi(0, t) = Q_\xi(1, t) = 0, \]  

and Dirichlet BCs

\[ Q(0, t) = Q_0(t) \in W^{1,2}(0, \infty), \quad Q(1, t) = Q_1(t) \in W^{1,2}(0, \infty). \]

In order to deal with strong, sufficiently smooth solutions of the uncertain PDE (46)–(48) (or (49) instead of (48)) in the open loop when no control input is applied, the uncertain term, which appears in the right-hand side of (46), is supposed to satisfy the same condition (6).

The control task is to make the scalar field \( Q(\xi, t) \) follow a given reference \( Q_r(\xi, t) \) such that

\[ Q_r \in L_2(0, \infty), \quad Q_{\xi\xi} \in L_2(0, 1). \]

Clearly, the reference \( Q_r \) should also be selected in accordance with the chosen BCs, namely, Neumann BCs

\[ Q_{\xi\xi}(0, t) = Q_{\xi\xi}(1, t) = 0 \]

or Dirichlet BCs

\[ Q_r(0, t) = Q_0(t), \quad Q_r(1, t) = Q_1(t). \]

Once the control task is achieved the deviation variable

\[ x(\xi, t) = Q(\xi, t) - Q_r(\xi, t) \]

will be driven to zero by the designed feedback control. The treatment will be preliminarily addressed in subsection 4.1 for the heat equation (46) with \( \psi(\xi, t) = 0 \), i.e., without the disturbance source term. Then, in subsection 4.2, the stabilization problem for the general perturbed system model (46) will be considered.
4.1. Finite-time control of the unperturbed heat equation. Let us consider system (46) with $\psi(\xi,t) = 0$, and let $Q_r(\xi,t)$ be selected according to (50)–(51) (or (52) instead of (51)) as the reference profile for $Q(\xi,t)$. The dynamics of the error variable (53) are governed by

$$x_t(\xi,t) = \theta_1 x_{\xi\xi}(\xi,t) + u(\xi,t) - Q_r(\xi,t) + \theta_1 Q_{r_{\xi\xi}}(\xi,t),$$

and the associated BCs are either of Neumann type

$$x_\xi(0,t) = x_\xi(1,t) = 0$$

or Dirichlet type

$$x(0,t) = x(1,t) = 0.$$

The following distributed power-fractional (DPF) control strategy is suggested in order to stabilize in finite time the tracking error dynamics (54) (with the BCs (55) or (56)) associated with the unperturbed heat equation:

$$u(\xi,t) = Q_r(\xi,t) - \theta_1 Q_{r_{\xi\xi}}(\xi,t) - \lambda_1 \frac{x(\xi,t)|x(\xi,t)|^\alpha}{\|x(\cdot,t)\|_2}, \quad 0 < \alpha < 1, \quad \lambda_1 > 0.$$

Note that the above controller incorporates a feedforward component $u_{ff}(\xi,t) = Q_r(\xi,t) - \theta_1 Q_{r_{\xi\xi}}(\xi,t)$ which depends on the reference $Q_r(\xi,t)$. Note also that the knowledge of $\theta_1$ can be dispensed with when the function $Q_{r_{\xi\xi}}$ is identically zero (this happens, for instance, when the reference $Q_r$ is either a constant or a function of the time variable only). By virtue of (50), the combined feedback/feedforward control algorithm (57) turns out to be well defined.

The next well-known lemma is stated to be used subsequently. Lemma 2 can be viewed as a particular case of the Hölder integral inequality

**Lemma 2** (see [1]). Consider an arbitrary real coefficient $p \geq 1$, and let $q(\cdot,t) \in L_p(0,1)$. Then the following inequality holds:

$$\left[ \int_0^1 |q(\xi,t)|^p d\xi \right]^{\frac{1}{p}} \leq \int_0^1 |q(\xi,t)|^p d\xi.$$

For later use, recall that system (54)–(56) is referred to as globally finite-time stable if it is globally asymptotically stable and any solution $x(\xi,t)$ to this system escapes to zero in finite time, and, furthermore, starting from that moment, it remains in the origin forever [16].

The following result is in force.

**Theorem 2.** Consider the tracking error dynamics (54) associated with the unperturbed diffusion equation, along with the BCs (55) or (56). Then, the DPF control strategy (57) guarantees the global finite-time stability of the error dynamics (54).

**Proof.** Consider the Lyapunov functional

$$V_1(t) = \frac{1}{2} \int_0^1 x^2(\xi,t) d\xi$$
whose time derivative along the strong solutions of system (54) is given by

\[ \dot{V}_1(t) = \int_0^1 x(\xi, t) x_t(\xi, t) d\xi \]

\[ = \int_0^1 x(\xi, t) \left( \theta_1 x_{\xi\xi}(\xi, t) + u(\xi, t) - Q_{\xi\xi}(\xi, t) + \theta_1 Q_{\xi\xi}(\xi, t) \right) d\xi \]

\[ = \theta_1 \int_0^1 x(\xi, t) x_{\xi\xi}(\xi, t) d\xi - \lambda_1 \int_0^1 x(\xi, t) \frac{x(\xi, t)^{a}}{\|x(\cdot, t)\|_2} d\xi \]

\[ = -\theta_1 \int_0^1 x^2(\xi, t) d\xi - \lambda_1 \int_0^1 |x(\xi, t)|^{2+\alpha} d\xi \]

\[ = -\theta_1 \int_0^1 x^2(\xi, t) d\xi - \lambda_1 \frac{\int_0^1 (x(\xi, t))^{2+\alpha} d\xi}{\|x(\cdot, t)\|_2} \]

\[ (60) \]

By virtue of Lemma 2, (60) can be further manipulated as follows:

\[ \dot{V}_1(t) \leq -\theta_1 \int_0^1 x^2(\xi, t) d\xi - \lambda_1 \left[ \int_0^1 (x^2(\xi, t))^{1+\alpha} d\xi \right] \]

\[ \leq -\theta_1 \int_0^1 x^2(\xi, t) d\xi - \lambda_1 \left[ \int_0^1 x^2(\xi, t) d\xi \right]^{\frac{1+\alpha}{2}} \]

\[ = -\theta_1 \int_0^1 x^2(\xi, t) d\xi - \lambda_1 [2V_1(t)]^{\frac{1+\alpha}{2}} \leq -\lambda_1 [2V_1(t)]^{\frac{1+\alpha}{2}}. \]

\[ (61) \]

As is well known (see, e.g., [16, Lemma 4.3], the resulting inequality (61) with the parameter \( \alpha \in (0, 1) \) ensures the monotone finite-time convergence to zero of \( V_1(t) \), thereby yielding the global finite-time stability of the unperturbed error dynamics (54).

**Remark 3.** The feedback part of controller (57) can be viewed as the infinite-dimensional extension of the finite-dimensional power fractional controller

\[ u_{fb}(t) = -\lambda_1 |x(t)|^\alpha \text{sign}(x(t)), \quad 0 < \alpha < 1, \quad \lambda_1 > 0. \]

An alternative, more straightforward infinite-dimensional extension

\[ u_{fb}(\xi, t) = -\lambda_1 |x(\xi, t)|^\alpha \text{sign}(x(\xi, t)), \quad 0 < \alpha < 1, \quad \lambda_1 > 0, \]

of the feedback control component (62) could also be used to stabilize the error dynamics. However, in this case only global asymptotic stability could be proved using the same Lyapunov functional (59).

### 4.2. Robust control of the heat equation.

Let us now consider the perturbed diffusion equation (46) including the uncertain disturbance field \( \psi(\xi, t) \). The class of admissible disturbances is specified by the following restrictions.

**Assumption 2.** There exist a priori known constants \( M \) and \( M_\xi \) such that the following restrictions hold uniformly beyond the origin \( \|x(\cdot, t)\|_2 = 0 \) in the state space \( L_2(0, 1) \):

\[ \|\psi(\cdot, t)\|_2 \leq M, \quad \|\psi_\xi(\cdot, t)\|_2 \leq M_\xi \quad \forall t \geq 0, \]

\[ (64) \]
Comment on Assumption 2. Since Remark 1 applies here as well, restriction (65) determines a region of admissible disturbance \( \psi \) in terms of a state-dependent constraint on its time derivative \( \psi_t \). With this interpretation, an admissible disturbance has a time derivative which is not necessarily vanishing as \( \|x(\cdot,t)\|_2 \to 0 \) because the norm in the right-hand side of the disturbance restriction (65) remains unit (indeed, \( \|x(\cdot,t)\|_2 = 1 \)). Particularly, a finite-dimensional counterpart of (65) would not impose any further restrictions on admissible disturbances in addition to the norm boundedness (64). It is worth noticing, however, that in the adopted infinite-dimensional treatment, the disturbance restriction (65) plays an important role and it can be skipped no longer.

The dynamics of the error variable (53) are now

\[
x(t,\xi) = \theta_1 x_{\xi\xi}(\xi, t) + u(\xi, t) - Q_r(\xi, t) + \theta_1 Q_{\xi\xi}(\xi, t) + \psi(\xi, t), \quad \theta_1 > 0,
\]

and the associated BCs are still given by (55) or (56).

In order to stabilize the error dynamics we propose a dynamical distributed controller defined as follows:

\[
u(t, \xi) = -W_1 \frac{x(\xi, t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi, t), \quad (56)
\]

which can be understood as a feedforward control action (depending on the a priori known reference signal \( Q_r(\xi, t) \)) plus a distributed version of the finite-dimensional “supertwisting” 2-SM controller [6, 12]. For ease of reference, the distributed super twisting controller (67)–(68) will be abbreviated as DSTW.

The solution concept of the perturbed diffusion equation (46)–(48) subject to the discontinuous control strategy (67)–(68) is defined in the same generalized sense [16] as that of the discontinuous wave PDE (2)–(4), (16). Definitions 2 and 3 are straightforwardly extended to the present case, and Remark 2 applies here as well. The performance of the closed-loop system is analyzed in the next theorem.

THEOREM 3. Consider the perturbed diffusion equation (46) along with the BCs (55) or (56) and with the uncertain disturbance satisfying Assumption 2. Then, the distributed control strategy (67)–(68) with the parameters \( \lambda_1, \lambda_2, W_1, \) and \( W_2 \) selected according to

\[
W_1 > \max \{M_1, 2M\}, \quad \lambda_1 > \max \left\{ 2M, \frac{2M}{W_1} \right\}, \quad W_2 > 0, \quad \lambda_2 > 0
\]

guarantees the global asymptotic stability of the error dynamics (66) subject to the associated BCs (55) or (56).

Proof. Let us define the auxiliary variable

\[
\delta(t, \xi) = v(\xi, t) + \psi(\xi, t).
\]

System (66) with the control law (67)–(68) yields the following closed-loop dynamics:

\[
x(t, \xi) = \theta_1 x_{\xi\xi}(\xi, t) - \lambda_1 \sqrt{|x(\xi, t)|} \sgn(x(\xi, t)) - \lambda_2 x(\xi, t) + \delta(t, \xi), \quad (71)
\]

\[
\delta(t, \xi) = -W_1 \frac{x(\xi, t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi, t) + \psi(\xi, t).
\]
Consider the Lyapunov functional
\begin{equation}
V_2(t) = 2W_1 \sqrt{\int_0^1 x^2(\xi,t)d\xi} + W_2 \int_0^1 x^2(\xi,t)d\xi + \frac{1}{2} \int_0^1 \delta^2(\xi,t)d\xi + \frac{1}{2} \int_0^1 s^2(\xi,t)d\xi,
\end{equation}
inspired from the finite-dimensional treatment [13], where
\begin{equation}
s(\xi,t) = \theta_1 x_{\xi\xi}(\xi,t) - \lambda_1 \sqrt{[x(\xi,t)]} \text{sign}(x(\xi,t)) - \lambda_2 x(\xi,t) + \delta(\xi,t).
\end{equation}

Note that the above-defined variable \( s(\xi,t) \) coincides with the right-hand side of (71); namely, it coincides with the time derivative \( x_t(\xi,t) \). In order to simplify the notation, the dependence of the system variables from the space and time variables \( (\xi,t) \) is omitted from this point on. Let us rewrite the functional \( V_2(t) \) in compact form as follows:
\begin{equation}
V_2(t) = 2W_1 \|x\|_2 + W_2 \|x\|^2 + \frac{1}{2} \|\delta\|^2 + \frac{1}{2} \|s\|^2.
\end{equation}

The time derivative of \( V_2(t) \) is given by
\begin{equation}
\dot{V}_2(t) = 2W_1 \int_0^1 x_x dx + 2W_2 \int_0^1 x_x dx + \int_0^1 \delta_x dx + \int_0^1 ss_x dx.
\end{equation}
Let us evaluate the time derivative of the auxiliary signal \( s(\xi,t) \) along the strong solutions of (71)–(72)
\begin{equation}
s_t(\xi,t) = \theta_1 s_{\xi\xi}(\xi,t) - \frac{1}{2} \lambda_1 \frac{s(\xi,t)}{\sqrt{|x(\xi,t)|}} - \lambda_2 s(\xi,t) + \delta_t(\xi,t)
\end{equation}
\begin{equation}
= \theta_1 s_{\xi\xi} - \frac{1}{2} \lambda_1 \frac{s}{\sqrt{|x|}} - \lambda_2 s - W_1 \frac{x}{\|x\|_2} - W_2 x + \psi_t.
\end{equation}

Equation (76) can be rewritten in the form
\begin{equation}
\dot{V}_2(t) = 2W_1 \int_0^1 xsd\xi + 2W_2 \int_0^1 xsd\xi + \int_0^1 \delta_x d\xi + \int_0^1 ss_x d\xi.
\end{equation}

Now let us evaluate the time derivative (78) along the strong solutions of (71)–(72). Substituting (74) into (78) and rearranging the resulting inequality yields
\begin{equation}
\dot{V}_2(t) = \frac{2W_1}{\|x\|^2} \int_0^1 xsd\xi + 2W_2 \int_0^1 xsd\xi - \frac{W_1}{\|x\|^2} \int_0^1 \delta_x d\xi - W_2 \int_0^1 \delta_x d\xi + \int_0^1 \psi_t d\xi
+ \int_0^1 s \left( \theta_1 s_{\xi\xi} - \frac{1}{2} \lambda_1 \frac{s}{\sqrt{|x|}} - \lambda_2 s - W_1 \frac{x}{\|x\|_2} - W_2 x + \psi_t \right) d\xi
= \frac{2W_1}{\|x\|^2} \int_0^1 xsd\xi + 2W_2 \int_0^1 xsd\xi - \frac{W_1}{\|x\|^2} \int_0^1 \delta_x d\xi - W_2 \int_0^1 \delta_x d\xi + \int_0^1 \psi_t d\xi
+ \theta_1 \int_0^1 ss_{\xi\xi} d\xi - \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \lambda_2 \int_0^1 s^2 d\xi - \frac{W_1}{\|x\|^2} \int_0^1 xsd\xi - W_2 \int_0^1 xsd\xi
+ \int_0^1 \psi t d\xi.
\end{equation}
which can be manipulated as follows:

\[
\dot{V}_2(t) = -\frac{W_1}{\|x\|_2} \int_0^1 x(\delta - s)d\xi - W_2 \int_0^1 x(\delta - s)d\xi + \theta_1 \int_0^1 ss\xi\xi d\xi \\
- \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \lambda_2 \int_0^1 s^2 d\xi + \int_0^1 (\delta + s)\psi_t d\xi.
\]  
(80)

Notice that, by virtue of (74), one has

\[
\delta - s = \lambda_1 \sqrt{|x|} \text{sign}(x) + \lambda_2 x - \theta_1 x_{\xi\xi},
\]
(81)

\[
\delta + s = 2s + \lambda_1 \sqrt{|x|} \text{sign}(x) + \lambda_2 x - \theta_1 x_{\xi\xi}.
\]
(82)

Due to this, (80) can further be manipulated as follows:

\[
\dot{V}_2(t) = -\frac{W_1 \lambda_1}{\|x\|_2} \int_0^1 x\sqrt{|x|} \text{sign}(x)d\xi - \frac{W_1 \lambda_2}{\|x\|_2} \int_0^1 x^2 d\xi + \frac{W_1 \theta_1}{\|x\|_2} \int_0^1 xx\xi\xi d\xi \\
- W_2 \lambda_1 \int_0^1 x\sqrt{|x|} \text{sign}(x)d\xi - W_2 \lambda_2 \int_0^1 x^2 d\xi \\
+ W_2 \theta_1 \int_0^1 xx\xi\xi d\xi + \theta_1 \int_0^1 ss\xi\xi d\xi \\
- \frac{1}{2} \lambda_1 \int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \lambda_2 \int_0^1 s^2 d\xi + 2 \int_0^1 s\psi_t d\xi \\
+ \lambda_1 \int_0^1 \sqrt{|x|} \text{sign}(x)\psi_t d\xi + \lambda_2 \int_0^1 x\psi_t d\xi \\
- \theta_1 \int_0^1 xx\psi_t d\xi.
\]  
(83)

Integrating by parts the term \( \int_0^1 xx\xi\xi d\xi \) and considering the BCs (55) or (56), it follows that

\[
\int_0^1 xx\xi\xi d\xi = -\int_0^1 x^2 d\xi + x(1,t)x\xi(1,t) - x(0,t)x\xi(0,t) = -\int_0^1 x^2 d\xi.
\]  
(84)

By applying the same manipulations to \( \int_0^1 ss\xi\xi d\xi \) it can be written that

\[
\int_0^1 ss\xi\xi d\xi = -\int_0^1 s^2 d\xi + s(1,t)s\xi(1,t) - s(0,t)s\xi(0,t).
\]  
(85)

Since, by virtue of (71), (74), \( s \) is the time derivative of \( x \), the same relationship holds between the respective BCs. It implies that the BCs of \( s(\xi, t) \) are all homogeneous, which in turn implies that

\[
\int_0^1 ss\xi\xi d\xi = -\int_0^1 s^2 d\xi.
\]  
(86)

It is worth noting that by virtue of (84)–(86) all integral terms appearing in (83) are negative definite except those depending on the disturbance term \( \psi_t(\xi, t) \).
also that only the time derivative $\psi_t$ of the original disturbance $\psi$ affects the right-hand side of (83), which is why no direct restrictions on $\psi$ are required by Assumption 2.

Clearly, the disturbance-depending terms appearing in (83) are sign indefinite. Some upper bounds to those terms are now derived by simple application of the Hölder and Minkowski integral inequalities [1], by considering Assumption 2, and by exploiting the well-known inequality [20]

$$\|x\|_2 \leq \|x\|_2,$$

(87)

$$2 \left| \int_0^1 s\psi_t d\xi \right| \leq 2 \int_0^1 |s|\sqrt{|\psi_t|} \sqrt{|\psi_t|} \frac{d\xi}{|x|} = 2 \int_0^1 |s|\sqrt{|\psi_t|} \frac{d\xi}{|x|} \leq M \int_0^1 s^2 \frac{d\xi}{|x|} + \int_0^1 \psi_t \sqrt{|x|} d\xi,$$

(88)

$$\lambda_2 \left| \int_0^1 x\psi_t d\xi \right| \leq \lambda_2 \left[ \int_0^1 x^2 d\xi \right]^{1/2} \left[ \int_0^1 \psi_t^2 d\xi \right]^{1/2} \leq \lambda_2 M \|x\|_2,$$

(89)

$$\theta_1 \left| \int_0^1 x\xi\psi_t d\xi \right| = \theta_1 \left[ \int_0^1 x\xi\psi_t d\xi \right]^{1/2} \left[ \int_0^1 \psi_t^2 \frac{d\xi}{|x|} \right]^{1/2} \leq \theta_1 M \|x\|_2 \|x\|_2 = \theta_1 M \|x\|_2 \|x\|_2,$$

(90)

Rewriting (83) in the more compact form yields

$$\dot{V}_2(t) = -W_1 \lambda_2 \|x\|_2 - W_2 \lambda_2 \|x\|_2 - W_1 \theta_1 \frac{\|x\|_2}{\|x\|_2} - W_2 \theta_1 \|x\|_2 - \lambda_2 \|s\|_2 - \lambda_2 \|s\|_2$$

$$- W_2 \lambda_1 \int_0^1 |x|^{3/2} d\xi + \frac{1}{2} \lambda_1 \int_0^1 s^2 d\xi \frac{1}{\|x\|_2} + \frac{W_1 \lambda_1}{\|x\|_2} \int_0^1 x \sqrt{|x|} \text{sign}(x) d\xi$$

$$+ 2 \int_0^1 s \psi_t d\xi + \lambda_1 \int_0^1 \sqrt{|x|} \text{sign}(x) \psi_t d\xi + \lambda_2 \int_0^1 x \psi_t d\xi - \theta_1 \int_0^1 x \psi_t d\xi.$$

(91)

Taking into account (88)–(90) it turns out that

$$\dot{V}_2(t) \leq -W_1 \lambda_2 \|x\|_2 - W_2 \lambda_2 \|x\|_2 - W_1 \theta_1 \frac{\|x\|_2}{\|x\|_2} - W_2 \theta_1 \|x\|_2 - \lambda_2 \|s\|_2 - \lambda_2 \|s\|_2$$

$$- W_2 \lambda_1 \int_0^1 |x|^{3/2} d\xi + \frac{1}{2} \lambda_1 \int_0^1 s^2 d\xi \frac{1}{\|x\|_2} + \frac{W_1 \lambda_1}{\|x\|_2} \int_0^1 x \sqrt{|x|} \text{sign}(x) d\xi$$

$$+ M \int_0^1 s^2 \frac{d\xi}{\sqrt{|x|}} + \int_0^1 \psi_t \sqrt{|x|} d\xi + \lambda_1 \int_0^1 \sqrt{|x|} \text{sign}(x) \psi_t d\xi$$

$$+ \lambda_2 M \|x\|_2 + \theta_1 M \|x\|_2 \|x\|_2,$$

(92)
which can be easily manipulated as follows:

\[
\dot{V}_2(t) \leq -\lambda_2(W_1 - M)\|x\|_2 - W_2\lambda_2\|x\|_2^2 - \theta_1(W_1 - M_\xi)\frac{\|x\|_2^3}{\|x\|_2}
\]

\[
- W_2\theta_1\|x\|_2^2 - \lambda_2\|s\|_2^2 - \theta_1\|s\|_2^2
\]

\[
- W_2\lambda_1 \int_0^1 \frac{|x|^{3/2}}{\sqrt{|x|}} d\xi - \frac{1}{2}(\lambda_1 - 2M) \int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \int_0^1 \sqrt{|x|} \left[ W_1\lambda_1 \frac{W_1}{2\|x\|_2} |x| - \psi_t \right] d\xi
\]

\[
- \lambda_1 \int_0^1 \sqrt{|x|} \left[ W_1 \frac{W_1}{2\|x\|_2} |x| - \psi_t \right] d\xi.
\]

By virtue of Assumption 2 and taking into account the tuning condition \(69\), the conditions

\[
W_1 > M, \quad W_1 > M_\xi, \quad \lambda_1 > 2M, \quad W_1\lambda_1 > 2M, \quad W_1 > 2M
\]

on the tuning parameters are met. It follows that all integral terms in the right-hand side of \(93\) are negative definite. Since the Lyapunov functional \(75\) is radially unbounded, the global asymptotic stability of the closed-loop system \(55\) (or \(56\) instead of \(55\)), \(66\)–\(69\) is thus established. This concludes the proof. \(\square\)

5. Simulation results—heat equation. Consider the perturbed heat equation \(46\) with homogeneous Neumann-type BCs as in \(55\). Let us assign the unit value to the thermal conductivity parameter \(\theta_1\). A constant set-point \(Q_r = 20\) is considered. The ICs are set to \(95\)

\[
Q(\xi,0) = \cos(2\pi\xi).
\]

In TEST 3 we preliminarily considered the unperturbed diffusion equation \(54\) and the DPF control \(57\) in Theorem 2. The gain \(\lambda_1\) is set to 10 and the power degree \(\alpha\) is set to 0.5. Figure 2(left) reports the attained solution. It is seen that the solution field \(Q(\xi,t)\) does converge to the set-point. Next, in TEST 4, let us introduce a space-varying disturbance term

\[
\psi(\xi,t) = 100\sin(2\pi\xi),
\]

and let us apply again the DPF control \(57\) to the perturbed system. According to the theoretical analysis made in Theorem 2, the closed-loop system under the control law \(57\) is guaranteed to be finite-time stable only with no perturbations acting on the system. Figure 2(right) reports the attained solution which, as expected, features a steady state error between the actual and desired behavior whose shape replicates the disturbance profile.

Now let us consider the distributed dynamical controller \(67\)–\(68\). Since the considered disturbance does not depend explicitly on the time variable, its derivatives \(\psi_t\) and \(\psi_{t\xi}\) are identically zero, which means that \(M = M_\xi = 0\). Therefore, in accordance with \(69\), any positive controller parameters \(\lambda_1, \lambda_2, W_1, W_2\) will stabilize the error dynamics. In TEST 5, controller \(67\)–\(68\) has been simulated using the parameters \(\lambda_1 = \lambda_2 = W_1 = W_2 = 5\). The solution is reported in Figure 3(left). Figure 3(right) depicts the control input \(u(\xi,t)\), which is a smooth function of both time and space, as expected.
Let us evaluate the \( L_2 \) norm bounds \( M \) and \( M_\xi \) of the derivatives \( \psi_t \) and \( \psi_\xi \), which are involved in the controller tuning inequalities (69):

\[
\psi_t(\xi,t) = 200\pi \sin(2\pi\xi)\cos(2\pi t) \implies \|\psi_t\|_2 = \frac{200\pi}{\sqrt{2}} \cos(2\pi t) \implies M = \frac{200\pi}{\sqrt{2}} \approx 444,
\]

\[
\psi_\xi(\xi,t) = 400\pi^2 \cos(2\pi\xi)\cos(2\pi t) \implies \|\psi_\xi\|_2 = \frac{400\pi^2}{\sqrt{2}} \sin(2\pi t) \implies M_\xi = \frac{400\pi^2}{\sqrt{2}} \approx 888.
\]

Then the controller gains are set to

\[
W_1 = 900, \quad \lambda_1 = 900, \quad W_2 = 10, \quad \lambda_2 = 10.
\]

Figure 4 depicts the solution \( Q(\xi,t) \) in TEST 6. As compared with TEST 5, the transient time is now much shorter due to the large values of the adopted tuning parameters \( \lambda_1 \) and \( W_1 \).
6. Conclusions. The so-called twisting and supertwisting 2-SM control algorithms are extended to globally asymptotically stabilize uncertain wave and, respectively, heat equations under Dirichlet and Neumann boundary conditions. The proposed infinite-dimensional treatment retains the main robustness features against non-vanishing disturbances similar to those possessed by its finite-dimensional counterpart. Although the present investigation is confined to the case where distributed sensing and actuation are available, the extension to pointwise and, particularly, boundary sensing and actuation appears to be possible. This, however, needs further investigation. Finite-time convergence of the proposed distributed twisting and supertwisting algorithms is among other actual problems to be tackled within the present framework. As a first step in this direction, which is completely new for infinite-dimensional systems, the finite-time convergence of an appropriate extension of the Bhat–Bernstein power-fractional algorithm is established for the unperturbed heat equation.

REFERENCES


