Finite-Time Converging Jump Observer for Switched Linear Systems with Unknown Inputs

F.J. Bejarano\textsuperscript{a}, A. Pisano\textsuperscript{b}, E. Usai\textsuperscript{b}

\textsuperscript{a}National Autonomous University of Mexico, Engineering Faculty, Division of Electrical Engineering, UNAM, México
\textsuperscript{b}University of Cagliari, Department of Electrical and Electronic Engineering, Italy

Abstract

This work addresses the state observation problem for a class of switched linear systems with unknown inputs. The proposed high-order sliding-mode observer provides a finite-time converging estimate of the continuous system’s state vector in spite of the presence of unknown inputs. The design procedure, which assumes the knowledge of the discrete state of the switched system, is based on the principles of disturbance decoupling and hierarchical observer output injection. In order to cope with the switching nature of the plant under observation, jumps in the observer state space are enforced intentionally. The implementation of an additional observer allows for the reconstruction of the unknown inputs, which may be important in the framework of fault detection. Numerical examples illustrate the effectiveness of the suggested technique.

\textsuperscript{✩}The work of F.J. Bejarano was partially supported by the Fondo de Cooperación Internacional en Ciencia y Tecnología UE-México (FONCICYT) grant no.93302. A. Pisano and E. Usai gratefully acknowledge the financial support from the FP7 European Research Projects “PRODI - Power plants Robustification by fault Diagnosis and Isolation techniques”, grant no.224233.

\texttt{Email addresses: javbejarano@yahoo.com.mx (F.J. Bejarano), pisano@diee.unica.it (A. Pisano), eusai@diee.unica.it (E. Usai)}
Keywords: Switched systems, sliding mode observer, unknown inputs

1. Introduction

1.1. Antecedents and motivations

The problem of state observation for systems with unknown inputs has been widely studied during the last two decades. The majority of the existing observers requires that the dimension of the unknown input vector does not exceed the number of measurable output variables. Additional special conditions are also met (see, e.g., Hautus (1983)) that often turn out to be very restrictive. Sliding mode observers (see, e.g. Barbot et al. (2002), Xiong and Saif (2001)) are widely used due to their attractive features: a) insensitivity with respect to a class of unknown inputs; b) possibility of using the equivalent output injection principle for the unknown input identification (see, e.g. Edwards et al. (2000)). More recently, a new generation of observers based on second-order sliding-mode (2-SM) algorithms has been suggested (see Shtessel et al. (2003), Bartolini et al. (2003), Pisano and Usai (2004), and Davila et al. (2005)). Finite-time convergence, along with robustness and unknown input identification capabilities, are the main characteristics of 2-SM based observers.

Molinari (Molinari (1976)) proposed a recursive algorithm dealing with the construction of a series of matrices that allows for the calculation of the weakly unobservable subspace in the last step of the algorithm. The matrices involved in the Molinari procedure are suitably used in the framework of the present work as disturbance-decoupling transformation matrices. The Molinari algorithm can also be interpreted in terms of recursively ap-
plied transformations and differentiation of the system’s output. Thus, if
the system is “strongly observable” the state can be reconstructed through a
recursive procedure of linear transformation and output differentiation. Such
an algorithm is the basis of the hierarchical observer proposed in Bejarano
et al. (2007).

In the literature it can be found only few papers related to observer design
for switched systems. Controllability and observability issues for switched
systems without unknown inputs are addressed in Suna and Geb (2005). In
Alessandri and Coletta (2001) linear time-invariant switched dynamics were
considered, and a switched version of the conventional Luenberger observer
was suggested. The problem of finding a common Lyapunov function for
the switching dynamics of the error system is studied via LMI techniques. In
Pettersson (2006) it was proposed to reset the observer state at the switching
times, which guarantees the boundedness of the observation error. A step by
step sliding mode observer is designed in Saadaoui et al. (2006) and Barbot
et al. (2007) for a special class of nonlinear switched systems. None of the
above work studied the problem in the presence of unknown inputs. Up to
the authors’ knowledge, the present work is the first manuscript where the
problem of observer design for a general class of switched linear systems with
unknown inputs is tackled.

1.2. Main contributions

We address and solve the finite time observation problem for a class of
switched linear systems with unknown inputs and dwell time for the first
switching only.

Under the assumption of strong observability for all the systems’ modes
of operation, and knowing the switching sequence (discrete state) of the switching system, we provide, both, the:

- finite time exact reconstruction of the system state, and the
- finite time exact reconstruction of the unknown inputs

The key tools employed in the paper and the main novelties with respect to Bejarano et al. (2007, 2009) are:

- explicit differentiator tuning formulas that allow to achieve an arbitrarily fast convergence of the observer, and
- a state jump (resetting) procedure applied to some observer internal variables preserving the correct reconstruction of the continuous state and unknown input across the switching instants.

2. Problem Formulation

Consider the uncertain linear switched system:

\[ \dot{x}(t) = A_{\lambda(t)}x(t) + Dw(t), \quad x(t_i^+) = x(t_i^-) \]

\[ y(t) = Cx(t) \]  

(1)

where \( x(t) \in \mathbb{R}^n \) is the system state, \( w(t) \in \mathbb{R}^q \) is the vector of the unknown inputs, \( y(t) \in \mathbb{R}^p \) is the vector of measurable outputs, and \( \lambda(t) : [0, \infty) \rightarrow \{1, 2, ..., Q\} \) is the “commutation function” (or “discrete state”) that sets the current system dynamics among the \( Q \) possible modes of operation represented by the characteristic matrices \( A_1, A_2, ..., A_Q \). Function \( \lambda(t) \) undergoes discontinuities at the time instants \( t_i (i = 1, 2, ...) \). Denote the time instant \( t_i \) as the \( i \)-th switching instant, and the interval \( T_i \equiv [t_{i-1}, t_i) \) as the \( i \)-th switching interval (with \( t_0 = 0 \) being the initial time). The continuous state vector does not exhibit jumps across switching instants according to
(1), with $t_i^−$ and $t_i^+$ are the instant just before and just after the $i$-th mode switching.

We assume that $\lambda(t)$ is known (a priori or in real-time) and we let $A_1, A_2, \ldots, A_Q, C,$ and $D$ be known constant matrices of appropriate dimension, with $\text{rank}(C) = p,$ and $\text{rank}(D) = q.$ The problem addressed in the paper is that of reconstructing in finite time the continuous system state $x(t)$ and the unknown input vector $w(t)$ in spite of the switched nature of the considered dynamics.

Throughout the paper it is assumed that:

A1. The matrix triplets $(A_i, C, D), i = 1, 2, \ldots, Q,$ are strongly observable

A2. There exists a constant $w^+$ such that $\|w(\cdot)\| \leq w^+.$

A3. The system state evolves into a known arbitrarily large invariant compact domain $D.$

A4. The first switching occurs at the time $t_1 \geq \varepsilon$, where $\varepsilon$ is a known constant.

The fulfillment of above assumptions A1-A4 is enough for building an observer that reconstruct the continuous state of the switched system (1). In order to reconstruct the unknown input as well, an additional separate observer will be designed, which requires the next additional assumption on the unknown input vector time derivative

A5. There exists a constant $w^+_d$ such that $\|\dot{w}(\cdot)\| \leq w^+.$

2.1. Comments on the given assumptions

Consider the standard linear time-invariant (i.e., non switched) system

\[ \dot{x}(t) = Ax(t) + Dw(t), \quad y(t) = Cx(t) \quad (2) \]
System (2) (or, equivalently, the matrix triplet \((A, D, C)\)) is called strongly observable if, for any initial condition \(x(0)\) and for any unknown input \(w(t)\), the condition \(y(t) = 0\) for all \(t \geq 0\) implies that \(x(t) = 0\) for all \(t \geq 0\).

All the possible dynamics are here required to meet the strong observability property, thus the requirement A1 could be understood as a **generalized strong observability condition** for the switched system (1). It is clear that assumption A1 is a necessary structural condition for the reconstruction of the state vector of the switching dynamics (1), and cannot be dispensed with. Assumption A1 can be equivalently reformulated by requiring that the matrix triplets \((A_i, C, D)\) do not possess zeros (see Bejarano et al. (2007)).

Assumptions A2 and A3 are due to the convergence conditions of the super-twisting differentiation algorithm Levant (1998), several instances of which are embedded in the proposed observer. Both assumptions could be partially relaxed by using alternative robust differentiation methods (e.g. the global differentiators presented in Pisano and Usai (2007); Levant (2007))

That generalization has been addressed in Bejarano et al. (2008) for the non switching dynamics (2). Global differentiators are, however, much more complicated to implement (see also Remark 2), and, for the sake of simplicity, we shall limit the present investigation to the case when assumptions A2 and A3 hold true.

Assumption A3 can be understood in terms of the considered switching system being bounded-input-bounded-state (BIBS) stable with respect to the unknown input \(w\). It implies that a constant \(x^+ = \sup_{x \in D} \|x\|\) can be found such that \(\|x(t)\| \leq x^+\).

Assumption A4 allows us to tune the observer scheme so that the corre-
sponding state estimation transient, which has finite duration, will terminate before that the first switching occurs. In other words, the continuous system state will be reconstructed starting from some time moment \( \tilde{t} \leq t_1 \leq \varepsilon \) (see Lemma 1). From that moment, as proven in the Theorem 1, the exact estimation of the state will be preserved across the next switching time instants, and no dwell-time constraint is put on the next switchings. This is made possible by the fact that the switching sequence \( \lambda(t) \) is supposed to be known in the present paper. It appears to be a very challenging problem to achieve the state estimation and unknown input reconstruction under the simultaneous effect of unknown mode switchings and unknown inputs. We will devote next research efforts to generalize the present result in that direction.

The approach could be also extended to the more general cases of: i) Presence of a known input term \( Bu(t) \), and ii) Time varying \( C_{\lambda(t)} \) and \( D_{\lambda(t)} \) matrices, by following nearly the same lines of reasoning but with a much more complicated mathematical treatment. Since the more general settings would not bring new ideas we prefer to keep the simplified problem statement in the present paper.

The implementation of our approach could be problematic for large scale systems of very large order \( n \). There is however an intriguing opportunity. Some decomposition is usually made in the large scale systems in terms of interconnected subsystems of lower order. In some cases, it can happen that the “coupling terms” (namely the interconnection terms between the different subsystems) may fulfill the structural requirements allowing to treat them as unknown inputs entering the dynamics of the generic subsystem. This
would make feasible a computationally efficient parallel implementation of the presented observer.

Assumption A5, that is needed only for reconstructing the unknown input, could be also relaxed by embedding the global second order sliding mode algorithms (see Pisano and Usai (2007); Levant (2007)) into the separate observer that reconstructs the unknown input.

2.2. Notation

For any vector \( z = [z_1, z_2, ..., z_q] \in \mathbb{R}^q \) and scalar number \( \alpha \in \mathbb{R} \) denote

\[
\text{sign}(z) = [\text{sign}(z_1), \text{sign}(z_2), ..., \text{sign}(z_q)]^T \\
|z|^\alpha = \text{diag}(|z_1|^\alpha, |z_2|^\alpha, ..., |z_q|^\alpha)
\]

(3)

For any matrix \( F \in \mathbb{R}^{r \times q} \) having rank \( F = h \), define \( F^\perp \in \mathbb{R}^{r-h \times r} \) as a matrix such that \( F^\perp F = 0 \) and rank \( F^\perp = r-h \). We also define a projection operator \( E_\alpha(z) : \mathbb{R} \times \mathbb{R}^q \rightarrow \mathbb{R}^q \) that will be useful in expressing in compact form the proposed observer. Let vector \( z \in \mathbb{R}^q \) be a function of time, \( z = z(t) \). The operator is defined as follows:

\[
E_\alpha(z(t)) = \begin{cases} 
0 & t < \alpha \\
z(t) - z(\alpha) & t \geq \alpha
\end{cases}
\]

(4)

3. Hierarchical 2-SM Observation for non-switching systems

Let us recall the solution proposed in Bejarano et al. (2007) for observing in finite time the state of system (2).

**Step 0.** Consider the output \( y = Cx \) and define \( M_{A,1} = C \).

**Step 1.** Consider the transformed output

\[
\xi_1(t) = (CD)^\perp y(t)
\]

(5)
and its derivative
\[
\frac{d}{dt} \xi_1 = \frac{d}{dt} (CD)^\perp y(t) = (CD)^\perp CAx(t) \tag{6}
\]

Construct the extended vector
\[
z^{(1)} = \begin{bmatrix} \frac{d}{dt} \xi_1 \\ y \end{bmatrix} = \begin{bmatrix} (CD)^\perp CA \\ C \end{bmatrix} x = M_{A,2} x \tag{7}
\]
where \(M_{A,2}\) is implicitly defined.

**Step k.** Consider the transformed output
\[
\xi_k(t) = (M_{A,k} D)^\perp z^{(k-1)}(t) \tag{8}
\]
and its derivative
\[
\frac{d}{dt} \xi_k = \frac{d}{dt} (M_{A,k} D)^\perp z^{(k-1)}(t) = (M_{A,k} D)^\perp M_{A,k} Ax(t) \tag{9}
\]

Construct the extended vector
\[
\begin{bmatrix} \frac{d}{dt} \xi_k \\ y(t) \end{bmatrix} = \begin{bmatrix} (M_{A,k} D)^\perp M_{A,k} A \\ C \end{bmatrix} x(t) \equiv M_{A,k+1} x(t) \tag{10}
\]
with implicit definition of \(M_{A,k+1}\).

It is known that if the system is strongly observable then \(\text{rank } M_{A,n} = n\).

It may also happen that \(\text{rank } M_{A,l} = n\) for some \(l \leq n\). This guarantees the possibility of recovering the state \(x\) through the relationship
\[
x(t) = M^+_{A,l} \begin{bmatrix} \frac{d}{dt} \xi_{l-1}(t) \\ y(t) \end{bmatrix} \tag{11}
\]
with \(M^+_{A,l} := (M^T_{A,l} M_{A,l})^{-1} M^T_{A,l}\). Thus, the state estimation can be solved if we are able to reconstruct, in finite time, the quantity \(M_{A,l} x\), with \(l\) being the least integer number such that \(\text{rank } M_{A,l} = n\).
3.1. Procedure for the recovery of $M_{A,l}x(t)$

According to (11) the state can be recovered by a linear combination of the output and its derivatives independently of the unknown inputs. In this paper we proposed to use the super twisting algorithm as a substitute of a differentiator. For recovering $M_{A,l}x(t)$ one needs to recover, hierarchically, $M_{A,k}x(t)$ for $k = 1, \ldots, l-1$. The first vector $M_{A,1}x(t)$ is already known since $M_{A,1} = C$ which means that $M_{A,1}x(t) = y$. The recovery of $M_{A,2}x(t)$ will be based on the design of a sliding surface $s^{(1)}$ and its corresponding output injection $v^{(1)}$ using the “super-twisting” algorithm (see Levant (1993)). The components of $v^{(1)}$ are defined as

$$
\dot{v}^{(1)} = -\lambda_{1,1} |s^{(1)}|^{1/2} \text{sign } s^{(1)} + \bar{v}^{(1)} \\
\dot{\bar{v}}^{(1)} = -\lambda_{1,2} \text{sign } s^{(1)}
$$

(12)

with the variable $s^{(1)} = v^{(1)} - \xi_1(t)$, and the transformed output $\xi_1$ defined in (5). The dimension of the vector $v^{(1)}$ is the same as the dimension of $s^{(1)}$, and this is equal to the number of rows of $(CD)^\perp$. In view of (2)-(5), and taking into account that $(CD)^\perp CD = 0$, the time derivative of $s^{(1)}$ is

$$
\dot{s}^{(1)}(t) = \dot{v}^{(1)}(t) - (CD)^\perp CAx(t).
$$

Now, with the appropriate selection of $\lambda_{1,1}, \lambda_{1,2}$ (see, e.g. Levant (2003a)), there exists a finite reaching time $t_{d1}$ such that the following equalities hold

$$
s^{(1)}(t) = \dot{s}^{(1)}(t) = 0, \quad t \geq t_{d1} \quad (13)
$$

From (12), it is clear that if $s^{(1)} = \dot{s}^{(1)} = 0$ then $\dot{v}^{(1)}(t) = d\xi_1/dt = \bar{v}^{(1)}$, which allows us to recover the vector $M_{A,2}x$ according to the following formula

$$
z^{(1)} = \begin{bmatrix} \bar{v}^{(1)} \\ y \end{bmatrix} = M_{A,2}x \quad t \geq t_{d1} \quad (14)
$$
Now, for recovering $MA_3x(t)$ we design a sliding surface $s^{(2)}$ and its corresponding output injection $v^{(2)}$

$$\dot{v}^{(2)} = -\lambda_{2,1} |s^{(2)}|^{1/2} \text{sign} (s^{(2)}) + \bar{v}^{(2)}$$

$$\dot{s}^{(2)} = -\lambda_{2,2} \text{sign} (s^{(2)})$$

(15)

with the variable $s^{(2)}$ given by the formula $s^{(2)}(t) = v^{(2)}(t) - \xi_2(t)$, and $\xi_2$ is defined in (8). The derivative of $s^{(2)}$ is $\dot{s}^{(2)}(t) = \dot{v}^{(2)}(t) - (MA_2D)^{\perp} MA_2Ax(t)$. With the correct selection of the scalar gains $\lambda_{2,1}$, $\lambda_{2,2}$ there exists a finite time $t_{d2} \geq t_{d1}$ such that the following equalities are true

$$s^{(2)}(t) = \dot{s}^{(2)}(t) = 0, \quad t \geq t_{d2}$$

(16)

Simple manipulations, similar to those made to derive formula (14), show that at $t \geq t_{d2}$ the following equality holds $\bar{v}^{(2)}(t) \equiv (MA_2D)^{\perp} MA_2Ax(t)$, thus we can write the following relationship for recovering $MA_3x(t)$.

$$z^{(2)}(t) := \begin{bmatrix} \bar{v}^{(2)}(t) \\ y \end{bmatrix} = MA_3x(t) \quad t \geq t_{d2}$$

(17)

We can follow the same procedure recursively to reconstruct $MA_kx(t)$, $k = 1, ..., l - 1$. Below we give the general design for the recovery of $MA_kx(t)$.

a) Design the output injection $\bar{v}^{(k)}$ at the $k$-th level as a “super-twisting” controller (see Levant (1993)):

$$\dot{v}^{(k)} = -\lambda_{k,1} |s^{(k)}|^{1/2} \text{sign} (s^{(k)}) + \bar{v}^{(k)}$$

$$\dot{s}^{(k)} = -\lambda_{k,2} \text{sign} (s^{(k)})$$

$$s^{(k)}(t) = v^{(k)}(t) - \xi_k(t)$$

(18)

with properly chosen coefficients $\lambda_{k,1}$ and $\lambda_{k,2}$ (Levant (2003a); Davila et al. (2005)). Thus, the design of the observer can be summarized by the following algorithm.
A. Compute the matrices $M_{A,1}$, $M_{A,2}$, ..., $M_{A,l}$ with $l$ being the least positive integer such that $\text{rank } M_{A,l} = n$.

B. Design $l - 1$ sliding surface vectors $s^{(k)}$ and output injections $v^{(k)}$ according to (18).

C. Recover the continuous state in finite time via the next relation

$$x(t) = M_{A,l}^+ \begin{bmatrix} \bar{v}^{(l-1)}(t) \\ y(t) \end{bmatrix}$$

(19)

4. Observation for switched systems

Here we study the observation problem for the switched system (1). It is the twofold aim of this section:

1. to derive observer tuning rules guaranteeing that the exact state estimation is already achieved within the first switching interval $[0, t_1)$

2. to preserve the exact state estimation during the successive switching intervals.

A “jump” observer, with some internal variable being intentionally reset at the switching times, will be proposed.

4.1. Observation during the first switching interval

It was previously shown how to compute, for a given triple $(A, D, C)$, the least integer $l$ such that $\text{rank } M_{A,l} = n$. In the actual case, this computation must be repeated separately for the $Q$ triples $(A_1, D, C)$, $(A_2, D, C)$, ..., $(A_Q, D, C)$, leading to the values $l_1, l_2, ..., l_Q$. At this point we define $\bar{l}$ as the largest $l_i$ value, i.e. $\bar{l} = \sup_{1 \leq i \leq Q} l_i$. Once $\bar{l}$ has been determined according to
the above procedure it turns out that the piecewise constant matrix $M_{A \lambda(t)}^+, I$ (which is constant during the switching intervals and abruptly changes at the switching instants) is well defined. The observer $\hat{x}(t)$ for system (1) can be defined by the following formula that generalizes eq. (19)

$$\hat{x}(t) = M_{A \lambda(t)}^+, I \begin{bmatrix} \bar{v}^{(I-1)}(t) \\ y(t) \end{bmatrix}$$

The recursive expression for calculating matrix $M_{A \lambda(t)}^+, I$ is as follows

$$M_{A \lambda(t),1} = C, \quad M_{A \lambda(t),k+1} = \begin{bmatrix} \left(M_{A \lambda(t),k} D\right)^\perp M_{A \lambda(t),k} A_{\lambda(t)} \\ C \end{bmatrix}, \quad k = 1, 2, \ldots$$

The signal $\bar{v}^{(I-1)}$ is recursively defined by

$$\dot{\bar{v}}^{(k)} = -\lambda_{k,1} |s^{(k)}|^{1/2} \text{sign} \left(s^{(k)}\right) + \bar{v}^{(k)}, \quad v^{(k)}(0) = 0 \quad (22)$$

for $k = 1, \ldots, I - 1$. Vectors $s^{(k)}$ should be designed by introducing some modifications as compared with the previous section. In order to understand the rationale of the associated modifications, let us think equations (22) as representative of $(I - 1)$ differentiators in cascade (see Fig. 1). The first differentiator ($k = 1$) receives as input signal $\xi_1(t) = (CD)^\perp y(t)$, and generates at the output signal $\bar{v}^{(1)}$. This is obtained by selecting the $s^{(1)}$ variable as follows

$$s^{(1)}(t) = v^{(1)}(t) - E_0(\xi_1(t))$$

After a finite time $t_{d1}$, the output $\bar{v}^{(1)}$ will be equal to the derivative of $\xi_1$. We want $t_{d1}$ to be less than $\varepsilon/(I - 1)$, i.e. that equality $\bar{v}^{(1)} = d\xi_1/dt$ is kept
for all \( t \in \left[ \frac{\varepsilon}{\bar{l} - 1}, t_1 \right] \). This will be guaranteed by appropriate tuning of the coefficients \( \lambda_{1,1} \) and \( \lambda_{1,2} \). However, the differentiator tuning formulas which allows to enforce an upper-bound to \( t_{d1} \) requires that \( s^{(1)}(0) = 0 \), which is the motivation for the application of the operator \( E_0(\cdot) \) to the first differentiator input \( \xi_1 \). Vectors \( s^{(k)} \) are designed according to the following formulas:

\[
\begin{align*}
  s^{(1)}(t) &= v^{(1)}(t) - E_0(\xi_1(t)) \\
  s^{(k)}(t) &= v^{(k)}(t) - E_{(k-1)}(\xi_k(t)) \\
  \xi_1(t) &= (CD)^{\perp} y(t) \\
  \xi_k(t) &= \left( M_{A_{M(t)},k} D \right)^{\perp} z^{(k-1)}(t) \\
  z^{(k-1)} &= \begin{bmatrix} \bar{v}^{(k-1)}(t) \\ y(t) \end{bmatrix}, \quad 1 < k \leq \bar{l} - 1
\end{align*}
\]

where \( E_\alpha(\cdot) \) is defined in (4). The output \( \bar{v}^{(2)} \) of the second differentiator is wanted to converge to the derivative of the input signal \( \xi_2 \) (which depends on \( \bar{v}^{(1)} \)) in a time \( t_{d2} \) not exceeding the value \( 2\varepsilon/(\bar{l} - 1) \). In order to enforce an upper-bound to \( t_{d2} \) through the appropriate tuning of the \( \lambda_{2,1} \)
and $\lambda_{2,2}$ coefficients, we require that $s^{(2)}(\varepsilon/(\bar{t} - 1)) = 0$ and the knowledge of an upper bound of $|$s^{(2)}(\varepsilon/(\bar{t} - 1))|. Therefore, the input of the second differentiator should be maintained to zero for all $t \in [0, \frac{\varepsilon}{\bar{t} - 1})$. This implies that $\dot{v}^{(2)} = v^{(2)} = 0$ in the same interval, and this is the motivation for the application of the operator $E_{\varepsilon/\bar{t} - 1} \cdot$ to the second differentiator input $\xi_2$, which guarantees, both, the fulfillment of condition $s^{(2)}(\varepsilon/(\bar{t} - 1)) = 0$ and the possibility to calculate an upper-bound of $|\dot{s}^{(2)}(\varepsilon/(\bar{t} - 1))|$ in the form $|s^{(2)}(\varepsilon/(\bar{t} - 1))|$. And so forth for $k > 2$. The hierarchical convergence mechanism outlined above is formalized by means of the next Lemma.

**Lemma 1.** Consider system (1) during the first switching interval $T_1 = [0, t_1) \subset [0, \varepsilon)$, and the observer (20)-(24) with the parameters $\lambda_{k,1}$ and $\lambda_{k,2}$ selected according to

$$\lambda_{k,1} > \frac{(1 + \theta)(\mu + 1)}{(1 - \theta)\sqrt{\mu - 1}} \sqrt{2L_k}, \quad \lambda_{k,2} = \mu L_k, \quad 0 < \theta < 1$$

(25)

for $k = 1, \ldots, \bar{t} - 1$, where

$$L_k = \max_{i \in Q} \left\| (M_{A_i,k} D)^\dagger M_{A_i,k} A_i \right\| \left( \|A_i\| x^+ + \|D\| w^+ \right)$$

$$\alpha_k = \max_{i \in Q} \alpha_{k,i}, \quad \alpha_{k,i} := \left\| (M_{A_i,k} D)^\dagger M_{A_i,k} A_i \right\|$$

$$\mu = 1 + \max_k \frac{\alpha_k (\bar{t} - 1)}{\varepsilon (1 - \theta) L_k x^+}$$

(26)

Then condition $x(t) = \hat{x}(t)$ is ensured at any time $t \in (\bar{t}, t_1)$, with $\bar{t} \leq \varepsilon$.

**Proof 1.** As demonstrated in Davila et al. (2005), we have that given the
\[ \dot{w}_1 = -\lambda_1 |s|^{1/2} \text{sign} (s) + w_2 \]
\[ \dot{w}_2 = -\lambda_2 \text{sign} (s), \quad s = w_1 - g(t), \]
\[ |\dot{g}(t)| \leq G_d, \quad w_1(0) = g(0), \quad w_2(0) = w_{20} \tag{27} \]
\[ \lambda_2 = \mu G_d, \]
\[ \lambda_1 > \frac{(1 + \theta)(\mu + 1)}{(1 - \theta)\sqrt{\mu - 1}} \sqrt{2G_d} \]

with
\[ \mu > 1 + \frac{|w_{20} - \dot{g}(0)|}{t^* (1 - \theta) G_d} \tag{28} \]

then condition \( w_2 \equiv \dot{g}(t) \) is achieved at any \( t \geq t^* \). Equations (27) and (28) are representative of a supertwisting differentiator whose parameters are set in order to enforce the (finite time) convergence of the differentiator in a time not exceeding \( t^* \). Conditions (22), (23) and the inequalities (26) turn out to be analogous to the respective equations and inequalities (27) and (28). Therefore, it yields that for the first differentiator \( (k = 1) \) the identity \( \bar{v}(1)(t) = \dot{\xi}_1(t) \) is ensured at any \( t \in \left[ \frac{\varepsilon}{l - 1}, t_1 \right] \). For the second differentiator we have that \( v(2) \left( \frac{\varepsilon}{l - 1} \right) = \bar{v}(2) \left( \frac{\varepsilon}{l - 1} \right) = E_{\mu \mu} (\xi_k (\frac{\varepsilon}{l - 1})) = 0 \). Therefore, with the tuning parameters given in (26), the identity \( \bar{v}(2)(t) = \dot{\xi}_2(t) \) is kept at any \( t \in \left[ \frac{2\varepsilon}{l - 1}, t_1 \right] \). By iterating the same considerations we get that \( \bar{v}(k)(t) = \dot{\xi}_k(t) \) for any \( t \in \left[ \frac{k\varepsilon}{l - 1}, t_1 \right] \). Lemma 1 is proven.

According to the above considerations, the proposed tuning conditions guarantee that the first differentiator will converge in a time \( t_{d1} \leq \varepsilon/(\bar{l} - 1) \), the second differentiator will converge in a time \( t_{d2} \leq 2\varepsilon/(\bar{l} - 1) \), and, by iteration, the last \( (\bar{l} - 1) \)-th differentiator will converge in a time \( t_{d_{\bar{l}-1}} \leq \varepsilon \), i.e., during the first switching interval according to \( \bar{v}^{(\bar{l}-1)}(t) = \dot{\xi}_{\bar{l}-1}(t) = \)
\[(M_{A,l-1}D)^\perp M_{A,l-1}Ax(t)\text{ for any } t \in [\varepsilon, t_1].\] Therefore, the observed state vector value \(\hat{x}(t)\) will converge to the actual one \(x(t)\) in a finite time \(T_{l-1} < \varepsilon\), and the identity \(\hat{x}(t) = x(t)\) is achieved for every \(t\) in the interval \([\varepsilon, t_1]\). The perfect derivatives estimate will be lost at the first switching instant due to the discontinuity in the high-order derivatives of the system output (caused by the hybrid system dynamics). To solve the problem we apply a "jump" to some observer internal variables. We show that if the actual output \(\bar{v}^{(k)}\) of each differentiator is suitably reset at the switching times, the observer will eventually achieve and keep the exact state estimate despite the abrupt changes in the system dynamics. At the time \(t^-_1\) we have
\[
\bar{v}^{(k)}(t^-_1) = \xi_k(t^-_1) = \left(M_{A\lambda(t^-_1)}D_{k+1} \right)^\perp M_{A\lambda(t^-_1)}A_{k+1} \hat{x}(t_1) \tag{29}
\]
while at time \(t^+_1\) we enforce the reset condition
\[
\bar{v}^{(k)}(t^+_1) = \left(M_{A\lambda(t^+_1)}A_{k+1} \right)^\perp M_{A\lambda(t^+_1)}A_{k+1}x(t_1) \tag{30}
\]
It is clear that, in general \(\hat{\xi}_k(t^+_1) \neq \hat{\xi}_k(t^-_1)\). Therefore, if \(\bar{v}^{(k)}\) does not jump at \(t^+_1\), we will have \(\bar{v}^{(k)}(t^+_1) \neq \hat{\xi}_k(t^+_1)\). Therefore, in order to maintain the exact convergence of the observer signal \(\bar{v}^{(k)}(t^+_1)\) must be reset to the value \(\hat{\xi}_k(t^+_1)\), which is known since at \(t = t_1\) the exact state estimate is already achieved, The overall resetting procedure is summarized in the next Theorem.

**Theorem 1.** The proposed observer (20)-(24) with the following state resetting rule to be applied at the switching instants
\[
\bar{v}^{(k)}(t^+_i) := \left(M_{A\lambda(t^+_i)}A_{k+1} \right)^\perp M_{A\lambda(t^+_i)}A_{k+1} \hat{x}(t^-_i) \tag{31}
\]
\( k = 1, 2, \ldots, \bar{l} - 1 \) and \( i = 1, 2, \ldots \) preserves the exact state estimation at any \( t \geq \varepsilon \).

**Proof 2.** From Lemma 1, we have that \( \hat{x}(t) = x(t) \) is achieved for \( t \in [\varepsilon, t_1) \). Furthermore, we have that \( s^{(k)}(t_1^-) = 0 \) and \( \bar{v}^{(k)}(t_1^-) = \hat{\xi}_k(t_1^-) \) \((k = 1, 2, \ldots, \bar{l} - 1)\). Since \( x(t) \) and \( s^{(k)}(t) \) are continuous functions, at the switching instant \( t_1^+ \) we get the identities

\[
\begin{align*}
\dot{s}^{(k)}(t_1^+) &= 0 \\
\ddot{s}^{(k)}(t_1^+) &= \bar{v}^{(k)}(t_1^+) - \dot{\xi}_k(t_1^+) \\
&= \bar{v}^{(k)}(t_1^+) - \left( M_{A\lambda(t_i^+)}^{k+1}D \right)^\perp M_{A\lambda(t_i^+)}^{k+1}x(t_i^-) \tag{33}
\end{align*}
\]

Thus, with the resetting of \( \bar{v}^{(k)}(t_1^+) \) given in (31), we get that \( \bar{v}^{(k)}(t_1^+) = \hat{\xi}_k(t_1^+) \), which implies that the sliding mode condition \( s^{(k)}(t) = \dot{s}^{(k)}(t) = 0 \) is kept across the switching time \( t_1 \) and, therefore, for all \( t \in [\varepsilon, t_2) \). Thus, condition \( \bar{v}^{(k)}(t) = \hat{\xi}_k(t) \) is ensured in the whole time interval \( t \in [\varepsilon, t_2) \). An inductive reasoning shows that the previous considerations can be made for the successive switching instants \( t_2, t_3 \) as well. Therefore, the exact observation condition \( x(t) = \hat{x}(t) \) will be maintained for all \( t \geq \varepsilon \). Theorem 1 is proven.

**5. Uncertainty Identification**

As soon as \( x(t) \) is exactly available it is possible to identify the unknown input vector \( w(t) \) according to the following scheme. Let us design the variable \( \bar{x} \) satisfying the following equation

\[
\dot{x}(t) = A_{\lambda(t)} \bar{x}(t) + Dv_w(t) \tag{34}
\]
Define a sliding variable $\sigma(t)$ in the form

$$\sigma(t) = D^+ (\ddot{x}(t) - \dot{x}(t))$$  \hspace{1cm} (35)$$

where $D^+ := (D^T D)^{-1} D^T$. Since $\ddot{x}(t) = x(t)$ for $t \geq \varepsilon$, and from (1), the derivative on time of $\sigma(t)$ is as follows:

$$\dot{\sigma}(t) = -w(t) + v_w(t) \text{ for } t \geq \varepsilon$$  \hspace{1cm} (36)$$

Then it can be designed $v_w$ in the following form

$$v_w = -1.5 \sqrt{w_d^+} |\sigma|^{1/2} \text{sign} (\sigma) + \hat{v}_w$$  \hspace{1cm} (37)$$

$$\dot{\hat{v}}_w = -1.1 w_d^+ \text{sign} (\sigma)$$  \hspace{1cm} (38)$$

Then according to standard convergence properties of the supertwisting algorithm there exist a finite time $\tilde{T}_w$ such that the following identities are achieved,

$$\sigma(t) \equiv \dot{\sigma}(t) \equiv 0$$  \hspace{1cm} (39)$$

for all $t \geq \tilde{T}_w$. Hence, from (36) and (39) we achieve in finite time the exact unknown input reconstruction, i.e.,

$$\hat{v}_w(t) \equiv w(t) \text{ for all } t \geq \tilde{T}_w.$$  \hspace{1cm} (40)$$

**Remark 1.** The presented method requires $w(t)$ to have uniformly bounded time derivative. If the time derivative of $w(t)$ is unbounded (i.e., signal $w(t)$ is bounded according to Assumption 1 but may exhibit some discontinuous jump) then the algorithm $v_w(t) = 1.1 w_d^+ \text{sign}(\sigma(t))$ could be adopted instead of (37). In that case, the unknown input could be recovered approximately via standard low pass filtering technique Utkin (1992).
6. Noise and discretization effects

The proposed scheme is based on the implementation of a cascade of “super-twisting” differentiators Levant (2003b), Levant (1998), according to the scheme depicted in the Figure 1. The measurement noise and the discretization effects due to the digital implementation of the differentiators will propagate along the “cascade” estimator, and it is the task of this Section to study the associated order of accuracy of the state and unknown input estimations with respect to, both, a measurement noise \( n(t) \) corrupting the output measurement and the sampling step \( T_s \) of the digitally-implemented differentiators.

Let \( n(t) \) be the additive noise corrupting the output measurements

\[
y_{\text{meas}}(t) = y(t) + n(t)
\]  

(41)

Let it be norm-bounded according to

\[
\| n(t) \| \leq \Sigma
\]

(42)

Let, also, \( T_s \) be the sampling period of the digital implementation algorithm approximating the suggested continuous-time observation scheme. As it was shown in Levant (2003b); Levant and Fridman (2004), the super-twisting differentiator algorithm (12), subject to the noisy transformed input (5), (41) and digitally implemented using sampled measurements with sampling step \( T_s \), is affected by the following differentiation error

\[
| \hat{v}^{(1)} - \dot{\xi}_1 | \leq \max(a_1 \sqrt{\Sigma}, b_1 T_s)
\]

(43)

with \( a_1 \) and \( b_1 \) being appropriate constants independent on \( \Sigma \) and \( T_s \). Such a differentiation error may be considered as an additive noise superimposed
to the signal entering the second differentiation stage (15). The resulting
differentiation error at the output of the second differentiation stage can be
then evaluated as Levant (2003b); Levant and Fridman (2004)

\[ | \hat{v}^{(2)} - \hat{\xi}_2 | \leq \max(a_2 \sqrt{\Sigma}, b_2 \sqrt{T_s}) \] (44)

with, again, properly defined constants \(a_2, b_2\). Iterating the same considera-
tions for the entire cascade of \(l - 1\) differentiator stages it yields that at the
output of the last stage the following error is present

\[ | \hat{v}^{(l-1)} - \hat{\xi}_{l-1} | \leq \max(a_{l-1} \Sigma^{2-\(l-1\)}, b_{l-1} T_s^{2-\(l-2\)}) \] (45)

Considering now the state reconstruction formula (19) the above inequal-
ity implies that the reconstructed state vector \(\hat{x}\) will differ from the actual
one according to

\[ \| \hat{x} - x \| \leq O(\Sigma^{2-\(l-1\)}) + O(T_s^{2-\(l-2\)}) \] (46)

The above formula has two main implications. First, it establishes the
accuracy improvement that can be obtained by reducing the sampling time
and/or by augmenting the precision of the output measurement devices. Sec-
ond, it establishes a property of “practical accuracy” of the suggested state
estimator, namely it is demonstrates that with sufficiently small values of \(T_s\)
and \(\Sigma\) it can be achieved any desired level of accuracy for the reconstructed
state.

A similar accuracy order evaluation can also be made concerning the
reconstructed unknown input vector \(w\). The super-twisting output injection
(37), with the sliding surface \(\sigma\) being corrupted by the “measurement error”
(46), is no longer capable of providing the ideal condition (39). Note that
the exact vanishing of $\dot{\sigma}$ led directly to the ideal estimation condition (40).
Actually, by taking into account (46) and by exploiting the same accuracy analysis as that made for the $(\bar{l} - 1)$ differentiation stages, it turns out the sliding surface derivative will be confined into the boundary layer

$$\|\dot{\sigma}\| \leq \max(a_w \Sigma^{2-\bar{l}}, b_w T_s^{2-(\bar{l}-1)})$$

which implies, in turns, that the actual and reconstructed unknown input vector $w$ and $\hat{w}$ will differ in the steady state according to

$$\|\hat{w} - w\| \leq O(\Sigma^{2-\bar{l}}) + O(T_s^{2-(\bar{l}-1)})$$

The above inequality has obviously the same twofold consequences as those previously discussed with reference to the condition (46). The validity of the relationships (46) and (48) will be verified in the simulation example section by performing appropriate comparative tests.

**Remark 2.** The accuracy order could be improved if, instead of using a cascade of first-order differentiators, a single differentiator of appropriate order Levant (2003b) is applied. In Bejarano and Fridman (2009) this method has been applied to observe the state of a class of non switched systems. The use of such an “arbitrary order” differentiator limits the propagation of the noise and discretization effects, as compared with a cascade of first order differentiators. However, there are two aspects that make it difficult to apply such a differentiator for the considered case. First, the resetting conditions (31) are difficult to be “translated” from the cascade scheme of first-order differentiators to the arbitrary-order differentiator. Second, no analytic results are yet available concerning the reaching-time of the arbitrary order differentiator. Therefore, it would be impossible to tune that differentiator in order to
ensure its convergence before the first switching. Then, embedding the arbitrary order differentiator into the proposed estimation scheme appears to be a challenging task that will be addressed in next works.

7. Simulation Examples

We present first an academic example and then a physically-motivated application example taken from the literature.

7.1. Example 1

Let us consider system (1) with order $n = 4, q = p = 2, Q = 2$ possible modes, and the following matrices, discrete state and unknown inputs

$$A_1 = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ (49)

$$A_2 = \begin{bmatrix} -2 & 1 & 1 & 0 \\ -3 & 1 & 0 & -2 \\ 2 & 0 & -3 & -1 \\ 1 & 0 & 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (50)

$$\lambda(t) = \begin{cases} 1 & 0 \leq t \leq 4 \\ 2 & t > 4 \end{cases}$$ (51)

$$w(t) = [\sin(t), \cos(t)]^T$$ (52)

The analysis of the triplets $(A_1, C, D)$ and $(A_2, C, D)$ reveals that assumption $A_1$ is fulfilled. Constants $w^+, x^+, \varepsilon$ and $w^+_{d}$ in the Assumptions
$A_2$-$A_5$ are set to the values $w^+ = 3$, $x^+ = 10$, $\varepsilon = 3$, $w_d^+ = 3$ by considering (49)-(52)

The matrices $M_{A_1,k}$ and $M_{A_2,k}$ are recursively computed for $k = 1, 2, \ldots$, according to the given formula (21). The rank of matrices $M_{A_1,2}$ and $M_{A_1,3}$ is 3 and 4, respectively. Therefore $l_1 = 3$. The same values are obtained with reference to the matrix $A_2$. Then $l_2 = 3$, and, consequently, $\bar{l} = 3$, which means that the cascade scheme in Figure 1 will contain two differentiator stages. The tuning parameters of the differentiators have been set as follows by means of the given formulas (25)-(26), with $\theta = 0.1$:

$$\lambda_{1,1} = 103, \quad \lambda_{1,2} = 236, \quad \lambda_{2,1} = 113, \quad \lambda_{2,2} = 274 \quad (53)$$

The observer (34)-(37) has been implemented to reconstruct the unknown input. The overall scheme has been implemented in Matlab-Simulink, and simulations have been made using the fixed-step Euler integration method with the step size $T_{s1} = 0.0001s$.

Fig. 2 shows the state trajectories for $x_1$ and $x_2$ together with their respective estimates. The finite-time convergence of the state estimates is apparent. It is also evident that the perfect estimation is preserved across the switching time $t_1 = 4$ thanks to the applied reset logic (31). The actual and reconstructed unknown inputs are displayed in Fig. 3. During a second test the resetting procedure (31) described in Theorem 1 has been not applied. Fig. 4 shows the obtained profiles for the actual and estimate state variables $x_1$ and $x_2$, which confirms that the suggested resetting logic is of fundamental importance.

In order to verify the correctness of (46) and (48), the estimation accuracy featured by the proposed observer is verified by performing two numerical
Figure 2: States trajectories $x_1$ and $x_2$ (solid) and theirs estimation (dashed).

Figure 3: Unknown inputs $w_1$ and $w_2$ (solid) and theirs estimation (dashed).
tests with different sampling periods. The previously presented simulations have been done by using the discrete measurement step $T_{s1} = 0.0001s$. The same simulation has been repeated using the value $T_{s2} = 10^{-6}s$. Let us refer as TEST1 and TEST 2 to the two simulations with $T_s = T_{s1}$ and $T_s = T_{s2}$, respectively. Since in the given example $\bar{l} = 3$, it is expected that, in the two tests, the following accuracy evaluations hold

$$\|\hat{x} - x\| \leq O(\sqrt{T_s})$$

$$\|\hat{w} - w\| \leq O(\sqrt{T_s})$$

Therefore, the expected improvement of accuracy in TEST 2 as compared to the TEST 1 is given by a factor $\sqrt{T_{s1}/T_{s2}} = 10$, as for the reconstruction of the state vector $x$, and $\sqrt{T_{s1}/T_{s2}} \approx 3.2$, as for the reconstruction of the unknown inputs. This is confirmed by the comparative analysis of the figures 5 and 6.
Figure 5: Observation error for $x_2$ and $w_2$ with a sampling time of $10^{-4}$ sec (TEST 1).

Figure 6: Observation error for $x_2$ and $w_2$ with a sampling time of $10^{-6}$sec (TEST 2).
7.2. Example 2

Let us consider the mass-spring system depicted in fig. 7, that was studied in De Witt et al. (2001). This model is intended to show the operation principle of a throttle valve for automotive applications. Let $x_1 = y$, $x_2 = \dot{y}$, be the valve plate position and velocity, $J_1$ and $J_2$ be the inertias of the moving parts, and $k_1$ and $k_2$ be the spring elastic coefficients. Let $u(t)$ be the driving force. Let also the system be subject to coulomb friction with magnitude $F_C$.

The elastic coefficient which is “seen” by the inertia $J_1$ is either $k_1$ or $k_2$, depending on the actual valve position according to

$$k(t) = k(x_1(t)) = \begin{cases} k_1 & x_1(t) \leq 0 \\ k_2 & x_1(t) > 0 \end{cases}$$

(56)

As discussed in De Witt et al. (2001), one typically has that $J_1 >> J_2$ then it can be assumed that $J = J_1 + J_2 \cong J_1$. Then the next state-space representation of the system dynamics is obtained

$$\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{k(t)}{J}x_1(t) + \frac{1}{J}u(t) + \frac{1}{J}(-F_C \text{sign} x_2(t) + \phi(t))
\end{align*}$$

(57)

where the term $\phi(t)$ is devoted to model the uncertainties and/or disturbances acting on the system.

The measurable output $y(t)$ is the position variable $x_1$, and the parameters $J$, $k_1$, $k_2$ and $F_C$ are supposed to be known. Since $x_2$ is supposed to be not available for measurements, we let the unknown input be

$$w(t) = -F_C \text{sign} x_2(t) + \phi(t)$$

(58)
We also set the driving force $u(t)$ to zero. Then, the switching dynamics of the system (56)-(57) can be represented in the form (1), with the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k_1}{J} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k_2}{J} \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(59)

The parameters are set as $k_1 = 2, k_2 = 3, J = 1, F_C = 1.5$, and the initial conditions are $x_1(0) = 3, x_2(0) = -2$. The disturbance $\phi(t)$ is selected as $\phi(t) = 2.5 \sin(t) + 0.3 \cos(10t)$.

The analysis of the triplets $(A_1, C, D)$ and $(A_2, C, D)$ reveals that assumption A1 is fulfilled. Assumptions A2-A4 hold as well, with the constants $w^+, x^+$, and $\varepsilon$ in the are set to the values $w^+ = 6, x^+ = 5, \varepsilon = 0.5$.

The matrices $M_{A_1,k}$ and $M_{A_2,k}$ are recursively computed for $k = 1, 2, \ldots$, according to the given formula (21). The suggested rank analysis of the obtained matrices reveals that $\bar{l} = 2$, i.e. only one differentiation stage is needed to reconstruct the state vector according to the scheme depicted in the Figure 1. Thus only one sliding variable $s = s^{(1)}$ should be designed.

Figure 7: Mass-spring system.
according to (22) and (24):

\[
\begin{align*}
\dot{s}^{(1)} &= v^{(1)} - x_1(t) + x_1(0) \\
v^{(1)} &= -\lambda_{1,1} \left| s^{(1)} \right|^{1/2} \text{sign} \ s^{(1)} + \bar{v}^{(1)}, \quad v^{(1)}(0) = 0 \\
\dot{\bar{v}}^{(1)} &= -\lambda_{1,2} \text{sign} \ s^{(1)}, \quad \bar{v}^{(1)}(0) = 0
\end{align*}
\]

The next step is to select \( \lambda_{1,1} \) and \( \lambda_{1,2} \). Taking into account the gain tuning rules (25)-(26) with \( \theta = 0.5 \) one derives that

\[ L_1 \approx 21, \quad \mu \approx 1.95 \]  

Thus we can make the next choice for the constant tuning parameters of (60)

\[ \lambda_{1,1} = 100, \quad \lambda_{1,2} = 41 \]  

There is a useful property concerning the reset condition (31), that specializes as follows for the considered example

\[ \bar{v}^{(1)} \left( t_i^+ \right) := \left( M_{A_{\lambda(t_i^+)}2D} \right)^{\perp} M_{A_{\lambda(t_i^+)}2A_{\lambda(t_i^+)}} \hat{x}(t_i^-) \]  

However, since \( M_{A_{\lambda(t_i^+)}} = I \), and \( D^{\perp} = \begin{bmatrix} 0 & 1 \end{bmatrix} \), then we obtain that

\[ \left( M_{A_{\lambda(t_i^+)}}2D \right)^{\perp} M_{A_{\lambda(t_i^+)}2A_{\lambda(t_i^+)}} \hat{x}(t_i^-) = \hat{x}(t_i^-) = \bar{v}^{(1)} \left( t_i^- \right) \]  

Condition (66) yields

\[ \bar{v}^{(1)} \left( t_i^+ \right) = \bar{v}^{(1)} \left( t_i^- \right) \]  

which means that \textbf{no reset should be applied} to the state of the considered observer at the switching time instants. This is obtained thanks to the special
structure of the considered system matrices. The overall scheme has been implemented in Matlab-Simulink, and simulations have been made using the fixed-step Euler integration method with the step size $T_s = 0.0001s$.

The discrete state $\lambda(t)$ is depicted in figure 8. The actual and estimated state $x_2$ is shown in figure 9. It can be seen that the state reconstruction is achieved well before the first switching time instant $t_1 \approx 0.7$.

The unknown input estimation is carried out by making a small modification to the scheme given in section 5. The modification is required since $w(t)$ has to meet the assumption A5, which is not the case for the formula
(58) due to the discontinuous Coulomb friction term. However, we should notice that the friction term $F_C \text{sign} \,(x_2)$ is perfectly known once the state reconstruction is achieved and $x_2$ becomes available and can be “injected” in the observer that reconstructs the unknown input. We consider the next modified version of (34)

\[
\dot{x} = A_\lambda(t)\dot{x} + D (-F_C \text{sign} \, \dot{x}_2 + \dot{v}_\phi)
\]  

(68)

Now the dynamics of the sliding variable $\sigma = D^+(\bar{x} - \hat{x})$ after that the condition $\dot{x}(t) = x(t)$ has been already achieved is

\[
\dot{\sigma}(t) = -\phi(t) + v_\phi(t) \quad \text{for } t \geq \varepsilon
\]  

(69)

We select $\phi^+_d = 10$ so that

\[
\left| \frac{d}{dt}(t) \phi(t) \right| \leq \phi^+_d
\]  

(70)

Then one can design $v_\phi$ according to the Supertwisting algorithm (Levant (1993)):

\[
\begin{align*}
    v_\phi &= -1.5 \sqrt{\phi^+_d \, |\sigma|^{1/2}} \text{sign} \,(\sigma) + \bar{v}_\phi \\
    \dot{\bar{v}}_\phi &= -1.1 \phi^+_d \text{sign} \,(\sigma)
\end{align*}
\]  

(71)  

(72)

obtaining, after a finite transient, the identities $\sigma(t) = \dot{\sigma}(t) = 0$, which imply that $\bar{v}_\phi(t) \equiv \phi(t)$ holds for sufficiently large $t$. The correct reconstruction of the disturbance $\phi(t)$ can be inspected in the Figure 10.

8. Conclusions

A new hierarchical approach to solve the observation problem for switched linear systems with unknown inputs is suggested. We propose a second-order
sliding mode observer under the main assumption of strong observability for all systems’ modes of operation. The proposed observation scheme, which assumes the knowledge of the discrete state, guarantees the finite time reconstruction of the continuous state before that the first mode switching occurs. A reset of some observer internal variable is enforced in order to preserve the exactness of the estimate across the successive mode switchings. The proposed observation scheme allows also to reconstruct the unknown inputs in finite time.

Several possible lines of improvements can be drawn for the presented result, by relaxing the given assumptions A2-A5 and by also considering the case when the discrete state is also uncertain and needs to be reconstructed.

References


Barbot, J., Djemai, M., Boukhobza, T., 2002. Sliding mode observers. In:


