Sliding mode control approaches to the robust regulation of linear multivariable fractional-order dynamics

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| Complete List of Authors: | Pisano, Alessandro; University of Cagliari, Dept. Electrical and Electronic Engineering 
Rapaic, Milan; University of Novi Sad, Faculty of Technical Sciences 
Jelicic, Zoran; University of Novi Sad, Faculty of Technical Sciences 
Novi Sad, Serbia, Faculty of Technical Sciences 
Usai, Elio; University of Cagliari, Dept. Electrical and Electronic Engineering |
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Sliding mode control approaches to the robust regulation of linear multivariable fractional-order dynamics


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Abstract

Sliding mode control approaches are developed to stabilize a class of linear uncertain fractional-order dynamics. After making a suitable transformation that simplifies the sliding manifold design, two sliding mode control schemes are presented. The first one is based on the conventional discontinuous first-order sliding mode control technique. The second scheme is based on the chattering-free second order sliding mode approach that leads to the same robust performance but using a continuous control action. Simple controller tuning formulas are constructively developed along the paper by Lyapunov analysis. The simulation results confirm the expected performance.

Keywords Fractional-order dynamics, sliding mode control

1 Introduction

Fractional-order systems, that is dynamical systems described using fractional (or, more precisely, non-integer) order derivatives and integrals are studied with growing interest in recent years. It has become apparent that a large number of physical phenomena can be described compactly by fractional-order systems theory [1, 2, 4, 3]. The pioneering applications of fractions calculus in control theory date back to the mid twentieth century [5]. In the nineties, Oustaloup proposed a non-integer robust control strategy named CRONE (Commande Robuste d’Ordre Non-Entier) [6]. Another well known fractional control algorithm is fractional PID (FPID or $PI^\lambda D^\mu$) controller introduced by Podlubny [7, 8].
Recently, optimal control theory has been generalized to incorporate models of fractional order [9, 10, 11, 12] and fractional calculus is penetrating other nonlinear control paradigms as well such as the model-reference adaptive control [13, 14, 15].

In the present paper we study the problem of asymptotically stabilizing a class of perturbed commensurate fractional-order linear time-invariant (FO-LTI) systems. Such systems are vastly studied in literature [13, 16, 17, 18, 19] and can be seen as the natural generalization of state-space models of conventional, integer-order, systems.

We allow unknown disturbances to enter the system dynamics, and for this reason we shall refer to the sliding mode (SMC) approach, a well-known nonlinear robust control technique. Conventional (says, “first-order”) sliding mode control (1-SMC) schemes make use of discontinuous control actions that give the closed-loop systems remarkable properties of robustness against significant classes of uncertainties and perturbations [20]. Although fractional calculus has been previously combined with sliding mode control in the controller design for conventional integer-order systems [21, 23], SMC has been applied to fractional-order systems only recently, see [22, 24]. In [22] perfectly known linear MIMO dynamics were studied, and a first-order sliding mode stabilizing controller was suggested, while in [24] nonlinear single-input fractional-order dynamics expressed in a form that can be considered as a fractional-order version of the chain-of-integrators “Brunowsky” normal form were studied. Sliding manifolds containing fractional-order derivatives were used in both works [22, 24].

The main drawback of sliding mode control is the so-called “chattering” phenomenon, namely the occurrence of undesirable high-frequency vibrations of the system variables which are caused by the discontinuous high-frequency nature of first-order sliding-mode control signals.

The second (and higher) order sliding mode control (2-SMC) approach is a
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recent and quite active area of investigation in the sliding mode control theory [25]. It was developed starting from the mid eighties [26, 27] to the main aim of improving the control accuracy and alleviating the undesired chattering effect by removing the control discontinuity while keeping similar properties of robustness analogous as those featured by the conventional first-order sliding mode approach.

In the present paper, the second-order sliding mode approach is applied for the first time to control uncertain fractional-order systems. A key point of the proposed approach is the selection of a special fractional-order sliding variable whose first-order total time derivative contains integer derivatives only, thus being manageable by standard sliding mode controllers. The transformation of the system equations to a special controllability normal form will be also developed, which permits a simple stability analysis of the sliding mode zero-dynamics.

To steer the system onto the fractional-order sliding manifold, two sliding mode techniques will be suggested. The first technique, based on the first-order sliding-mode approach, gives rise to a discontinuous high-frequency input to the system. The second technique, based on a modified version of the “super-twisting” second-order sliding mode control algorithm [26], gives rise to a continuous control input thereby removing the chattering effect.

The proposed controllers are very simple to implement as compared to alternative robust control approaches (e.g., adaptive control). The main contributions of this paper with respect to the related literature can be summarized as follows:

• inclusion of unknown matching perturbations in the multivariable fractional-order system equations

• a normal-form for linear multivariable perturbed fractional-order model
fulfilling the controllability assumption

- a new stability analysis for the zero-dynamics associated to the suggested fractional-order sliding manifold
- two control approaches for steering the system onto the fractional-order sliding manifold in finite-time
- simple and constructive controller tuning conditions

The outline of the paper is as follows. The next Section 2 recalls some preliminaries on fractional calculus. Section 3 contains the problem formulation. Section 4 presents the sliding manifold design procedure. Section 5 presents two sliding mode control approaches: the 1-SMC approach (subsect. 5.1) and the 2-SMC approach (subsect. 5.2) together with the respective Lyapunov-based stability analysis. Section 6 illustrates some simulation results, and the final section 7 draws some concluding remarks.

2 Preliminaries on Fractional Calculus

Fractional calculus (FC) is a remarkably old topic. Its origins can be traced back to the end of seventeenth century, to the famous correspondence between Marquise de L'Hospital and G. W. Leibnitz in 1695. Since than it has been addressed by many famous mathematicians, including Euler, Lagrange, Laplace, Fourier and others. However, in consequent centuries it remained a purely theoretical topic, with little if any connections to practical problems of physics and engineering. In recent decades, FC is found to be a valuable tool in many applied disciplines, ranging from mechanics and elasticity to control theory and signal processing. The first text devoted solely to fractional calculus is the book by Oldham and Spanier [28] published in 1974. Since then, numerous texts emerged, however the primary references used within this paper are the book
by Podlubny [7] and the recent one by Kilbas, Srivastava and Trujillo [29].

Several definitions of fractional operators appear in literature. In the current paper the so called Riemann–Liouville approach is adopted. The Riemann–Liouville fractional integral of order $\alpha \geq 0$ is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t x(\tau)(t-\tau)^{\alpha-1} \, d\tau$$  \hspace{1cm} (1)

where $x(t)$ is a scalar or a vector signal and $\Gamma(\alpha)$ is the Euler’s Gamma function

$$\Gamma(\alpha) = \int_0^\infty \nu^{\alpha-1} e^{-\nu} \, d\nu.$$  \hspace{1cm} (2)

For integer values of integration order $\alpha$ Riemann–Liouville fractional integral is equivalent to the classical $n$-fold integral. In fact, in such a case (1) reduces to the well known Cauchy formula

$$I^n x(t) = \frac{1}{(n-1)!} \int_0^t x(\tau)(t-\tau)^{n-1} \, d\tau.$$  \hspace{1cm} (3)

The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined as

$$D^\alpha x(t) = \frac{d^n}{dt^n} I^{n-\alpha} x(t)$$ \hspace{1cm} (4)

where $n$ is the smallest integer larger than $\alpha$, that is $n < \alpha \leq n - 1$. It can be proven, although this is not trivial, that for integer values of $\alpha$ fractional derivative coincides with the classical one. Within the current paper $\alpha \in (0, 1)$ is of primary interest. For such values of $\alpha$ the definition (4) becomes

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} \, d\tau$$ \hspace{1cm} (5)

To conclude this section, let us prove the following statement that will be used extensively in the sequel.
LEMMA 1: Consider a vector signal $z(t) \in \mathbb{R}^m$. Let $\alpha \in (0, 1)$. If there exists $t_1 < \infty$ such that

$$I^\alpha z(t) = 0 \quad \forall t \geq t_1$$  \hspace{1cm} (6)

then

$$\lim_{t \to \infty} z(t) = 0.$$  \hspace{1cm} (7)

PROOF. To prove the claim, first note that (6) is equivalent to $I^\alpha z(t) = a(t)$ where $a(t)$ is arbitrary function identically equal to zero for $t \geq t_1$. On the other hand, since the fractional derivative is the left inverse of the fractional integral [29], this is equivalent to saying that $z(t) = D^\alpha a(t)$, or

$$z(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{a(\tau)}{(t - \tau)\alpha} d\tau$$  \hspace{1cm} (8)

For large values of $t$, in fact for all $t \geq t_1$, this reduces to

$$z(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^{t_1} \frac{a(\tau)}{(t - \tau)\alpha} d\tau$$  \hspace{1cm} (9)

because $a(t)$ clips the upper limit of the integral. The time variable $t$ can now be seen as a parameter, and under mild conditions the differentiation can be introduced inside the integral, yielding

$$z(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^{t_1} a(\tau) \frac{\partial}{\partial t} \frac{1}{(t - \tau)\alpha} d\tau = \frac {-\alpha}{\Gamma(1 - \alpha)} \int_0^{t_1} a(\tau) \frac{1}{(t - \tau)^{\alpha+1}} d\tau$$  \hspace{1cm} (10)

Now, it is clear that

$$|z(t)| \leq \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^{t_1} |a(\tau)| \frac{1}{|t - \tau|^{\alpha+1}} d\tau$$  \hspace{1cm} (11)

and since the right hand side is zero in limit, $|z(t)|$ must be also. This concludes the proof. ■
3 Problem Formulation

Consider a fractional-order linear multivariable system affected by a matched unknown perturbation

\[ D^\alpha x(t) = Ax(t) + B(u(t) + d(t)) \quad 0 < \alpha < 1 \]  

(12)

where \( x(t) \in \mathbb{R}^n \) represents the “state” vector, which is supposed to be available for measurement, \( u(t) \in \mathbb{R}^m \) represents the input vector, \( A \) and \( B \) are the characteristic and control matrices, having appropriate dimensions, and \( d(t) \) is a bounded uncertain disturbance. The class of fractional-order dynamics (12) is called commensurate because all internal variables \( x(t) \) are differentiated with the same order \( \alpha \). In this paper, we consider only the commensurate case. However, this is not a major restriction, since a variety of fractional order models are in fact commensurate. For example, all models with fractional derivatives of rational order can be seen as commensurate, with \( \alpha \) equal to the reciprocal value of the least common multiple of denominators of all derivatives appearing in the model.

As noted earlier, (12) can be seen as a generalization of classical state-space model. However, fractional order systems are inherently infinite dimensional, and therefore the components of \( x(t) \) can not be seen as states of the considered systems. To emphasize this, and in accordance to [13, 16], the term vector-space model will be used in the sequel to denote (12). The components of \( x(t) \) will be denoted as the internal variables.

Let us introduce the following assumptions:

**Assumption A_1.** \((A, B)\) is a controllable pair, with matrix \( B \) being full rank \((\text{rank}(B) = m)\)

**Assumption A_2.** \(\|d(t)\| \leq d_M(t)\)

The control task is the asymptotic stabilization of the system (12).
We define the m-dimensional sliding manifold in the form

\[ \sigma = C I^{1-\alpha} x = 0 \]  

(13)

where \( \sigma \in \mathbb{R}^m \) and \( C \in \mathbb{R}^{m \times n} \) is a constant matrix. Such form for the sliding variable was already suggested in [22]. With respect to [22] we shall use a second-order sliding mode approach, we include unknown matching perturbations in the system equation, and we make a thorough and constructive controller design and stability analysis that was only partially addressed in [22].

From now on the design problem entails the following two steps. In the first step, described in Section 4, the sliding manifold is designed. In the second step, described in Section 5, the appropriate control input has been developed.

4 Sliding manifold design

This step concerns the design of the matrix \( C \) in order to assign a prescribed stable sliding mode dynamics.

The “sliding mode dynamics” is the dynamics of the original system after that it has been constrained to evolve on the sliding manifold \( \sigma = 0 \). Considering the special form (13) for the selected sliding manifold, the sliding mode dynamics is actually described by a fractional-order integro-differential system of the type

\[
\begin{align*}
D^{\alpha} x &= Ax + B(u + d) \\
\sigma &= C I^{1-\alpha} x = 0
\end{align*}
\]

(14)

with \( \sigma = [\sigma_1 \ \sigma_2 \ \ldots \ \sigma_m] \).

The standard approaches to sliding mode dynamics analysis for linear MIMO systems [20, 30] do not readily apply to the considered case because of the sliding variable \( \sigma \) do not contain \( x \) directly, but its fractional integral of order \((1-\alpha)\).

From the linearity of the fractional integral operator, the sliding manifold can be written in the form

\[
\sigma = I^{1-\alpha}(Cx) = 0
\]

(15)
The integral of order $(1 - \alpha)$ of every component of vector $Cx$ is steered to zero. This means, according to Lemma 1, that every component of vector $Cx$ tends to zero asymptotically starting from the moment at which $\sigma$ is identically zero.

The analysis of the sliding mode dynamics can then refer to the following system

$$
D^\alpha x = Ax + B(u + d) 
$$

(16)

$$
Cx = \eta(t) 
$$

(17)

where $\eta(t)$ is an asymptotically vanishing term. Let matrix $C$ be selected in such a way that the square matrix $CB$, of order $m$, be nonsingular.

Since $\text{rank}(B) = m$ there exist an invertible transformation matrix $T$ such that

$$
TB = \begin{bmatrix}
0 \\
B_2
\end{bmatrix} 
$$

(18)

where $B_2 \in \mathbb{R}^{m \times m}$ and $B_2$ is nonsingular. The transformed internal vector $z$ can be constructed as

$$
z = Tx 
$$

(19)

with $z = [z_1^T z_2^T]^T$, $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^m$, such that the transformed system dynamics is

$$
D^\alpha z_1 = A_{11} z_1 + A_{12} z_2 
$$

(20)

$$
D^\alpha z_2 = A_{21} z_1 + A_{22} z_2 + B_2(u + d) 
$$

(21)

$$
CT^{-1}z = \eta(t) 
$$

(22)

with the matrices $A_{ij}$ such that

$$
TAT^{-1} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} 
$$

(23)

Actually, the output map equation (22) of the transformed system represents an $m$-dimensional algebraic constraint involving the system “states” that
reduces the order of the sliding mode dynamics with respect to that of the original plant. Let us partition matrix $CT^{-1}$ as

$$
CT^{-1} = [C_1 \ C_2] \tag{24}
$$

in such a way that

$$
CT^{-1}z = C_1z_1 + C_2z_2 \tag{25}
$$

The assumption that the matrix product $CB$ is nonsingular implies that the matrix $C_2$ is nonsingular too [30]. One can rewrite the output equation as

$$
z_2 = -C_2^{-1}C_1z_1 + C_2^{-1}\eta(t) \tag{26}
$$

By considering (26) into the first of (22) it yields the following sliding mode dynamics governing equation

$$
\begin{align*}
D^\alpha z_1 &= (A_{11} - A_{12}C_2^{-1}C_1)z_1 + A_{12}C_2^{-1}\eta(t) \tag{27} \\
z_2 &= -C_2^{-1}C_1z_1 + C_2^{-1}\eta(t) \tag{28}
\end{align*}
$$

Thus, the criteria for selecting the $C$ matrix are as follows:
- $CB$ must be nonsingular
- $M = C_2^{-1}C_1$ must be such that the following dynamics is asymptotically stable

$$
D^\alpha z_1 = (A_{11} - A_{12}M)z_1 + A_{12}C_2^{-1}\eta(t) = \bar{A}z_1 + A_{12}C_2^{-1}\eta(t) \tag{29}
$$

with implicit definition of matrix $\bar{A} = A_{11} - A_{12}M$. Since signal $\eta(t)$ is asymptotically vanishing, the asymptotic stability properties of system (29) will be governed by its characteristic matrix $\bar{A}$ only. Once the asymptotic convergence to zero of vector $z_1$ is ensured by the appropriate selection of the matrix $M$, the successive asymptotic vanishing of vector $z_2$ trivially results from (28).

The conditions for the asymptotic stability of general linear fractional order dynamics are not yet fully understood. The following Lemma applies to
the considered class of linear, time invariant, commensurate fractional order systems.

**Lemma 2.** [18, 19] Consider system $D^\alpha z_1 = \bar{A} z_1$, with $\alpha \in (0, 1)$, and let $\lambda_i(\bar{A})$ ($i = 1, 2, ..., m$) be the eigenvalues of matrix $\bar{A}$. The system is asymptotically stable if and only if the following condition holds

$$|\arg(\lambda_i(\bar{A}))| > \frac{\alpha \pi}{2}, \quad i = 1, 2, ..., m \quad (30)$$

Thus, by Lemma 2, the matrix $M$ should be designed to place the eigenvalues of the matrix $\bar{A} = A_{11} - A_{12}M$ according to the restriction (30). The possibility of assigning the eigenvalues of the matrix $\bar{A} = A_{11} - A_{12}M$ is granted by the following Proposition.

**Proposition 1** [[30] p. 39] The matrix pair $(A_{11}, A_{12})$ is controllable if and only if the matrix pair $(A, B)$ is controllable.

It should be noted that the above design procedure, fixing $M = C_2^{-1}C_1$ only, does not uniquely determine matrix $C$. A computationally convenient way to set the matrices $C_1$ and $C_2$ is as follows:

$$C_2 = I_m \quad C_1 = M \quad (31)$$

which gives rise to

$$C = [M \quad I_m]T \quad (32)$$

where $I_m$ is the $m$-th order identity matrix.

**Remark 1.** A possible choice for the transformation matrix $T$ is

$$T = \begin{bmatrix} B^\perp \\ T_1 \end{bmatrix} \quad (33)$$

where $B^\perp$ is a matrix such that $B^\perp B = 0$ and $B^\perp$ is linearly independent of $B$, and $T_1$ is any matrix that makes $T$ and $T_1B$ nonsingular. A possible choice for $T_1$ is

$$T_1 = (B^T B)^{-1}B^T \quad (34)$$
Clearly, this choice guarantees the decomposition (18), with $B_2 = T_1B = I$

5 Control-input design

This step concerns the design of control schemes for steering the system (12) in finite time onto the sliding manifold (13). The task is not trivial due to, both, the presence of the unknown disturbance and the fractional-order nature of the system dynamics. By (4) it yields that

$$\dot{\sigma} = C \alpha x$$

In light of the plant equation (12) it follows that the dynamics of the sliding variable $\sigma$ is of integer order and uniform vector relative degree one [31].

$$\dot{\sigma} = CAx + CB(u + d)$$ (36)

Two sliding mode control approaches to the finite time stabilization of system (36) will be illustrated in the following subsections 5.1 and 5.2.

5.1 First-order sliding mode approach

The control vector can be selected as follows

$$u(t) = -(CB)^{-1} [CAx(t) - \rho \sigma(t) - (\kappa + \mu d_M(t))\text{sign}(\sigma(t))]$$ (37)

where

$$\text{sign}(\sigma) = [\text{sign}(\sigma_1) \text{ sign}(\sigma_2) \ldots \text{ sign}(\sigma_m)]^T$$ (38)

with the scalar parameters $\rho$ and $\mu$ such that

$$\rho > 0, \quad \kappa > 0 \quad \mu > \|CB\|$$ (39)

The finite time stability of system (36) with the control (37) is now proven by Lyapunov analysis. Let

$$V = \frac{1}{2} \sigma^T \sigma = \frac{1}{2} \|\sigma\|^2$$ (40)
be the Lyapunov function candidate. The derivative of $V$ along the trajectories of system (36) is

$$
\dot{V} = -\rho \sigma^T \sigma - (\mu d_M(t) + \kappa)\sigma^T \text{sign}(\sigma(t)) + \sigma^T CBd(t)
$$

(41)

The following chain of inequalities can be written by simple manipulations

$$
\dot{V} \leq -2\rho V - (\kappa + (\mu - \|CB\|)d_M(t))\|\sigma\|_1 \leq -2\rho V - \kappa_1 \sqrt{V} \quad \kappa_1 > 0
$$

(42)

By the comparison Lemma [31] it follow easily that $V(\sigma(t))$, and therefore $\sigma(t)$, globally converge to zero in finite time and reach the zero value at the time instant $T^* \leq 2[V(\sigma(0))]^{1/2}/\kappa_1$

5.2 Second-order sliding mode approach

The control vector $u = [u_1 \, u_2 \ldots \, u_m]$ is now expressed as follows

$$
u = (CB)^{-1}v$$

(43)

$$v = -CAx(t) - k_1\sigma - k_2|\sigma|^{1/2}\text{sign}(\sigma) + w$$

(44)

$$\dot{w} = -k_3\text{sign}(\sigma)$$

(45)

where the following notation is used to give (44)-(45) a more compact representation.

$$
|\sigma|^{1/2}\text{sign}(\sigma) = [\sqrt{|\sigma_1|}\text{sign}(\sigma_1) \sqrt{|\sigma_2|}\text{sign}(\sigma_2) \ldots \sqrt{|\sigma_m|}\text{sign}(\sigma_m)]^T
$$

(46)

Note that, unlike the first-order SMC law (37), the control law (43)-(46) defines a continuous input to the plant that avoids, or at least strongly attenuates, the chattering phenomenon.

A new assumption $A_3$ is introduced that replaces the former assumption $A_2$.

**Assumption A3.** A known constant $d_{Md}$ exists such that $\|\frac{d}{dt}d(t)\| \leq d_{Md}$
By dispensing with the Assumption A2 we now allow the components of the unknown disturbance vector \( d(t) \) to grow unbounded, but on the other hand we require them to be smooth signals, which was not necessary with the first-order sliding mode control approach.

The next Theorem 1 is proven, which constitutes the main result of the present paper.

**Theorem 1** Consider system (12) satisfying the Assumptions \( A_1, A_3 \), and the sliding manifold (13) with the \( C \) matrix designed according to the procedure given in the Section 4. Then, the control law (43)-(46), with the scalar parameters \( k_1, k_2, k_3 \) fulfilling the following tuning conditions

\[
 k_1 > 2\sqrt{\rho} \quad k_2 > 0 \quad k_3 > \rho \sqrt{\frac{k_1}{\rho}} \quad \rho = \|CB\|_{M_d} \quad (47)
\]

will steer the system (12) asymptotically to the origin.

**Proof of Theorem 1**

The closed loop system dynamics is obtained as follows considering (43) into (36)

\[
\begin{align*}
\dot{\sigma} &= CBd - k_1 \sigma - k_2 |\sigma|^{1/2} \text{sign}(\sigma) + w \\
\dot{w} &= -k_3 \text{sign}(\sigma)
\end{align*}
\]

Define the following new variable

\[
z = w + CBd
\]

and rewrite system (48) in the new \( \sigma - z \) coordinates

\[
\begin{align*}
\dot{\sigma} &= -k_1 \sigma - k_2 |\sigma|^{1/2} \text{sign}(\sigma) + z \\
\dot{z} &= -k_3 \text{sign}(\sigma) + CB\frac{d}{dt}d(t)
\end{align*}
\]

The dynamics of the variable pairs \( (\sigma_i, z_i) \in \mathbb{R}^2, i = 1, 2, ..., m \), are decoupled one each other, then to simplify the stability analysis it is convenient to refer to the decoupled systems independently

\[
\begin{align*}
\dot{\sigma}_i &= -k_1 \sigma_i - k_2 |\sigma_i|^{1/2} \text{sign}(\sigma_i) + z_i \quad i = 1, 2, ..., m \\
\dot{z}_i &= -k_3 \text{sign}(\sigma_i) + c_i B\frac{d}{dt}d(t)
\end{align*}
\]
with $c_i$ being the $i$-th row of matrix $C$. For the uncertain term $c_iB\frac{d}{dt}d(t)$ the following bound holds by virtue of assumption $A_3$

$$\left|c_iB\frac{d}{dt}d(t)\right| \leq \|CB\|d_Md \tag{52}$$

The dynamics (51)-(52) is a special case of the more general second-order dynamics studied in [32], Theorem 5. The same Lyapunov function as that used in [32] is considered:

$$V_i = 2k_3|\sigma_i| + \frac{1}{2}z_i^2 + \frac{1}{2}\left(k_1|\sigma_i|^{1/2}\text{sign}(\sigma_i) + k_1\sigma_i - z_i\right)^2 \tag{53}$$

which can be rewritten as follows

$$V_i = \xi^T H \xi \tag{54}$$

$$\xi = \begin{bmatrix} |\sigma_i|^{1/2}\text{sign}(\sigma_i) \\ \sigma_i \\ z_i \end{bmatrix}, \quad H = \begin{bmatrix} (4k_3 + k_2^2) & k_1k_2 & -k_2 \\ k_1k_2 & k_1^2 & -k_1 \\ -k_2 & -k_1 & 2 \end{bmatrix} \tag{55}$$

By evaluating the derivative of (54)-(55) along the trajectories of system (51)-(52), and considering the tuning rules (47), it can be found two positive constants $\gamma_1$ and $\gamma_2$ such that

$$\dot{V}_i \leq -\gamma_1V_i - \gamma_2\sqrt{V_i} \tag{56}$$

which easily implies, by simple application of the comparison Lemma, that all the $V_i$ Lyapunov functions, $i = 1, 2, ..., m$, tend to zero in a finite time, and the same holds for the vector $\sigma$. As shown in the Section 4, the finite time vanishing of the sliding vector variable $\sigma$ guarantees that all the $x(t)$ solutions of the uncertain system (12) will tend globally and asymptotically to zero. This proves the Theorem. ■

6 Simulation results

Distributed parameters processes, heat transfer in particular, constitute a rich area of application of fractional calculus. Recently, Melchior and coworkers [4], [4],
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considered a test bench involving long aluminum rod heated from one
of its sides, and showed a good agreement between the experimental data and
a commensurate fractional-order linear model of the system. The input \( u(t) \) to
such model is the thermal flux applied at one end of the rod, and the output is
the actual temperature at a prescribed section of the rod. The obtained model
is commensurate, and its vector-space formulation is as follows

\[
D^{0.5}x(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -0.0601251 & -0.42833
\end{bmatrix} \begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix} \quad \text{(57)}
\]

In the sequel, we will assume that all of the vector-space variables are accessible
for measurement and will use this model to test the control strategies discussed
previously. We also added a matching disturbance \( d(t) = D_0 + D_1 \sin(t) =
1 + \sin(t) \) like in (12) to test the robustness properties of the suggested sliding
mode controllers. The bounding terms \( d_M(t) \) and \( d_{Md} \) can be the selected as

\[
d_M(t) = D_0 + D_1 \quad d_{Md} = D_1 \quad \text{(58)}
\]

In order to design the vector \( C \) defining the sliding manifold the transformation
matrix \( T \) is computed first according to (33)-(34). It yields

\[
T = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{(59)}
\]

The resulting decomposition (23) yields

\[
A_{11} = \begin{bmatrix}
0 & 0 \\
-1 & 0
\end{bmatrix}, \quad A_{21} = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
-0.0601 & 0 \\
0 & -0.4283
\end{bmatrix} \quad \text{(60)}
\]

The matrix \( M \) that places the eigenvalues of matrix \( A_{11} - A_{12}M \) in the position
\([-2 -3]\), which are selected according to the stability condition (30) of Lemma 2,
is \( M = [5 \ -6] \), which can be easily computed via the Matlab “place” command.
Vector \( C \) defining the chosen sliding manifold is then derived according to (32)
as \( C = [6 \ 5 \ 1] \). Since \( CB = 1 \) the norm of \( CB \), which is involved in the controller

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tuning formulas, is given by
\[ \|CB\| = 1 \]  \hspace{1cm} (61)

The first and second order sliding mode control algorithms (37) and (43)-(44) have been implemented with the following parameter values, that are selected according to the respective tuning inequalities (39) and (47):

1-SMC: \[ \rho = 1, \quad \kappa = 0.1, \quad \mu = 1.1. \]  \hspace{1cm} (62)

2-SMC: \[ \rho = 1, \quad k_1 = 3, \quad k_2 = 1, \quad k_3 = 3.5 \]  \hspace{1cm} (63)

In all tests, the sampling period \( T_s = 0.0002s \) has been used. The fractional order dynamics is simulated by first computing the fundamental matrix \( \Phi(t) \), equal to the inverse Laplace transform of \( (s^{0.5}I - A)^{-1} \), where \( s \) is the Laplace variable and \( I \) is the unit matrix of appropriate size. The inverse Laplace transform was calculated using the series expansion method introduced by Atanacković et al. [34]. The vector-space response is then calculated according to
\[ x(t) = \Phi(t)I_0 + \int_0^t \Phi(t - \tau)Bu(\tau)d\tau \]  \hspace{1cm} (64)

with \( I_0 = \lim_{\alpha \to 0} D^{\alpha-1}x(t) \) being the initial condition (not equal to \( x(0) \)). In the following simulations \( I_0 \) was set to \([0.1 - 0.2 0.1] \).

The next Figure 1 shows the time evolution of the three internal variables with the first-order sliding mode control technique. The internal variable \( x_3 \) exhibits a noticeable chattering in the steady state. The reason is the discontinuous and high-frequency switching nature of the control input \( u \) which is shown in the Figure 2-left. Figure 2-right shows the time history of the sliding variable \( \sigma(t) \). The internal variables with the second-order sliding mode control technique are shown in the figure 3. It is apparent that the steady-state chattering oscillations are strongly attenuated as compared with the test with the first-order sliding mode control. The figure 4-left displays the control input \( u(t) \) which is now a **continuous** function of time. The time history of the sliding
The state variable with the first-order sliding mode control

Figure 1: The internal variables with the first-order sliding mode control.

variable \( \sigma(t) \) is reported in the figure 4-right. The tests confirm the good robustness properties of both the approaches and highlight the better performance featured by the second-order sliding mode control technique.

7 Conclusions

Two sliding mode techniques have been suggested in order to asymptotically stabilize a class of perturbed linear fractional-order dynamics. The first technique, based on the first-order sliding-mode approach, gives rise to a discontinuous high-frequency input to the system. The second technique, based on a modified version of the “super-twisting” second-order sliding mode control algorithm, gives rise to a continuous control input by “transferring” the high-frequency discontinuity to the time derivative of the actual control.

The core of the proposed approaches was the selection of a special fractional-order sliding variable whose first time derivative along the system trajectories contains integer derivatives only. The two approaches have been demonstrated via Lyapunov analysis, and simulations have been given to show their ef-
The control input $u(t)$ with the first-order sliding mode control

The sliding variable $\sigma$ with the first-order sliding mode control

Figure 2: The control input (left) and the sliding variable $\sigma$ (right) with the first-order sliding mode control.

The state variable with the second-order sliding mode control

Figure 3: The internal variables with the second-order sliding mode control.
Figure 4: The control input (left) and the sliding variable $\sigma$ (right) with the second-order sliding mode control.

There are multiple possible lines of improvement of the present results, that will be pursued in next works. More general classes of linear fractional-order systems, e.g. the non-commensurate ones, as well as some classes of nonlinear fractional-order dynamics will be studied first. Additional interesting problems to study could also be related to the output-feedback controller implementation.

References


