On the multi-input second-order sliding mode control of nonlinear uncertain systems

Alessandro Pisano*†

Department of Electrical and Electronic Engineering, University of Cagliari, Cagliari, Italy

SUMMARY

This note addresses the multi-input second-order sliding mode control design for a class of nonlinear multivariable uncertain dynamics. Among the most important peculiarities of the considered control problem, the considered sliding vector variable has a uniform vector relative degree \( k_2, k_2, \ldots, k_2/c_1 \) with respect to the vector control variable, and only the sign of the sliding vector and of its derivative are available for feedback. Additionally, the symmetric part of the state-dependent control matrix is supposed to be positive definite. Under some further mild restrictions on the uncertain system’s dynamics, a control algorithm that realizes a multi-input version of the ‘twisting’ second-order sliding mode control algorithm is suggested. Simple controller tuning conditions are derived by means of a constructive Lyapunov analysis, which demonstrates that the suggested control algorithm guarantees the semiglobal asymptotic convergence to the sliding manifold. Simulation results, which confirm the good performance of the proposed scheme and investigate the actual accuracy obtained under the discrete-time implementation effects, are given. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Sliding mode control (SMC) is a robust design method that is widely applied in controlling nonlinear uncertain processes [1–3]. High accuracy and robustness against matching uncertainties and disturbances, as well as simplicity of design and implementation, are the main positive features of SMC [4]. SMC design is, basically, a two-step procedure. First, a manifold in the state space (the ‘sliding manifold’) must be identified such that the system exhibits desired properties of stability when constrained to evolve on it. The second step is to find a controller (often a discontinuous one) steering the system trajectories towards the sliding manifold in finite time. The vanishing of the associated (vector) sliding variable is the mathematical condition representing the attainment of the sliding motion along the chosen sliding manifold.

In order to evaluate and compare different multivariable SMC approaches, it is of fundamental importance to take into account the assumptions on the high-frequency gain (HFG) matrix of the sliding variable dynamics, which is generally admitted to be uncertain in the SMC design framework. Let \( m \) be the dimension of the sliding and control vectors (say, \( \sigma \) and \( u \)). The HFG matrix of the sliding variable dynamics (say, \( G(\cdot) \)), possibly state dependent, will be therefore a square, uncertain, matrix of order \( m \).

The conventional (i.e., first-order) SMC theory offers a number of effective solutions to steer the system onto the sliding manifold when the sliding variable has a uniform vector relative degree...
\( \mathbf{r}_d = [1, 1, \ldots, 1] \) with respect to the control vector \([1, 2]\). In this case, the general form of the sliding variable dynamics can be represented as

\[
\dot{\mathbf{\sigma}} = \mathbf{A}(\mathbf{\cdot}) + \mathbf{G}(\mathbf{\cdot}) \mathbf{u}, \quad \mathbf{\sigma}, \mathbf{u} \in \mathbb{R}^m, \tag{1}
\]

with the uncertain HFG matrix \( \mathbf{G}(\mathbf{\cdot}) \) being globally nonsingular and with the ‘drift’ vector field \( \mathbf{A}(\mathbf{\cdot}) \) subject to appropriate growth restrictions. Note that the elements of \( \mathbf{A}(\mathbf{\cdot}) \) and \( \mathbf{G}(\mathbf{\cdot}) \) can be nonlinear functions of both the system state and time variables.

### 1.1. Literature analysis

With reference to the multivariable dynamics (1), let us review the main assumptions on the HFG matrix \( \mathbf{G}(\mathbf{\cdot}) \) that are met in the current literature. The usual assumption for the matrix \( \mathbf{G}(\mathbf{\cdot}) \) is that of being globally positive definite (GPD) \([1]\). Under this condition, several multi-input SMC approaches have been developed; among them, the relay-based SMC \([1, 2]\), the simplex-based SMC \([5]\), and the unit vector \([6]\) are worthy to mention. In \([1]\), a solution was suggested to deal with the case when \( \mathbf{G}(\mathbf{\cdot}) \) is not necessarily GPD, but its symmetrical part, \( 1/2 (\mathbf{G}(\mathbf{\cdot}) + \mathbf{G}(\mathbf{\cdot})^T) \), is actually GPD. If \( \mathbf{G}(\mathbf{\cdot}) \) is diagonally dominant, and all its principal minors have known sign, a special, so-called hierarchical (or ‘component-wise’) implementation of relay-based SMC can be made \([1, 7]\).

In \([8]\), the HFG matrix has been expressed in the form \( \mathbf{G}(\mathbf{\cdot}) = \mathbf{G}_{\text{nom}}(\mathbf{\cdot}) + \mathbf{\hat{G}}(\mathbf{\cdot}) \), with \( \mathbf{G}_{\text{nom}}(\mathbf{\cdot}) \) being a known nonsingular nominal matrix and \( \mathbf{\hat{G}}(\mathbf{\cdot}) \) being an uncertain matrix. The next positive definiteness condition was assumed

\[
\mathbf{I}_m + \frac{1}{2} \left[ \mathbf{\hat{G}}(\mathbf{\cdot}) \mathbf{G}_{\text{nom}}^{-1}(\mathbf{\cdot}) + \left( \mathbf{G}_{\text{nom}}^{-1}(\mathbf{\cdot}) \right)^T \mathbf{\hat{G}}(\mathbf{\cdot}) \right] > 0. \tag{2}
\]

The combination between sliding mode and adaptive control techniques led to substantial and important developments in the area. In \([9]\), a combined Variable Structure/Model Reference Adaptive Control (VS-MRAC) technique was suggested under the assumptions that the plant dynamics are linear time invariant (which implies that matrix \( \mathbf{G} \) should be constant) and that a known constant matrix \( \mathbf{S}_1 \) can be found such that \( \mathbf{S}_1 \mathbf{G} \) is positive definite. Such results have been later extended in \([10]\) where, among other improvements (e.g., including uncertain nonlinearities in the system equations), it was assumed to know a matrix \( \mathbf{S}_2 \) such that \( -\mathbf{S}_2 \mathbf{G} \) is Hurwitz, which is less restrictive.

Noticeably, in the framework of VS-MRAC, important achievements have been made concerning the admissible vector relative degree \( \mathbf{r}_d \). In \([11]\), the restriction of assuming \( \mathbf{r}_d = [1, 1, \ldots, 1] \) has been dispensed with by addressing the general case of the sliding variable dynamics having an arbitrary well-defined vector relative degree. The assumptions concerning the HFG matrix were the same as those made in \([10]\). In \([11]\), an effective combination between unit vector and output-feedback model-reference adaptive control concepts allows to asymptotically reject uncertain nonlinearities that grow linearly with respect to the unmeasurable state variables.

This paper addresses multivariable dynamics with state-dependent HFG matrix, which prevents the application of the VS-MRAC concepts and results, and vector relative degree \( \mathbf{r}_d = [2, 2, \ldots, 2] \). The sliding variable dynamics that shall be considered in the present work has the expression

\[
\dot{\mathbf{\sigma}} = \mathbf{A}(\mathbf{\cdot}) + \mathbf{G}(\mathbf{\cdot}) \mathbf{u}, \quad \mathbf{\sigma}, \mathbf{u} \in \mathbb{R}^m, \tag{3}
\]

in which \( \mathbf{A}(\mathbf{\cdot}) \) and \( \mathbf{G}(\mathbf{\cdot}) \) are state-dependent uncertain matrices and only the sign of the sliding vector components, and those of their time derivatives, are considered to be available for measurement.

For relative degree two systems, the second-order SMC (2-SMC) approach \([12, 13]\) appears to be particularly suited. The 2-SMC approach \([12–15]\) is an important generalization of the conventional (first-order) SMC. Such technique allows to stabilize the sliding variable dynamics (3) by acting, discontinuously, on the second derivative of \( \mathbf{\sigma} \) \([13–16]\). In relative degree one systems, this property allows to ‘transfer’ the discontinuity of the control algorithm to the time derivative of the actual plant control input (dynamic input extension), thereby obtaining, after integration, a continuous control input \([17–19]\). Alternatively, one can exploit the increase of relative degree (from one to two) by
including the first-order unmodeled actuator dynamics in the plant equations (in [20], an electrically controlled torque actuator was explicitly taken into account in the robot trajectory tracking problem). In [18], multi-input versions of the ‘suboptimal’ 2-SMC algorithm were presented, and the stabilization problem for system (3) was solved in the following two cases.

Case A. Matrix $G$ is positive definite; its entries are upper and lower bounded by known constants ($g_{ij} \leq |g_{ij}| \leq \overline{g}_{ij}$), and it is ‘sufficiently diagonally dominant’. The latter condition is expressed mathematically by the two following requirements:

$$|g_{ii}| > \sum_{j=1, j \neq i}^{m} |g_{ij}|, \quad \overline{g}_{ij} < \frac{3g_{ij}}{\overline{g}_{ii} + 4g_{ii}} \quad \forall i, j = 1, 2, \ldots, m.$$ 

Case B. Matrix $G$ is positive definite; its entries are upper and lower bounded by known constants, as previously, and a positive scalar is known such that the matrix $G + \mu I_m$ (where $I_m$ being the $m \times m$ identity matrix) is diagonally dominant. A rather complex solution, based on a dynamical auxiliary observer system, was suggested. Such an auxiliary observer behaves, in fact, as a real-time differentiator, and it is therefore sensitive against the measurement noise.

In [16], among other results on homogeneous discontinuous control, the stabilization problem for the dynamics (3) was addressed and solved in the simplified case of identity HFG matrix, that is, $G = I_m$. Clearly, this assumption greatly simplifies the control problem because there is no dynamic coupling between the control components. The control law proposed in this paper, and the rationale of the convergence proof, feature, however, close similarities with the technique presented in [16], Theorem 5.1.

Concerning the drift vector field $A\cdot$, in both works [16, 18], it was assumed to be globally norm-bounded by a known positive constant, that is,

$$\|A\cdot\| \leq \overline{A}.\quad (4)$$

In this paper, we are going to relax, or dispense with, most of the previously mentioned restrictions.

1.2. Contribution and structure of the paper

In the present work, with reference to nonlinear uncertain multi-input systems expressed in block normal form, an approach that avoids the use of any observer/differentiator and requires only the measure of the sign of the sliding vector and of its derivative is presented. Furthermore, we relax the condition (4) by considering a more general state-dependent upper bound to the norm of the drift vector field $A\cdot$. The HFG matrix $G\cdot$ is assumed to be uncertain, and it is allowed to be state dependent. The main requirement is made so that the eigenvalues of its symmetric part $G^s\cdot = \frac{1}{2}[G\cdot + G^T\cdot]$ are state-dependent positive-definite functions separated from zero and subjected to mild growth restrictions. In the present paper, the upper bounds to the norm of all system uncertainties can have an arbitrary rate of growth with respect to the system state. This includes systems featuring the finite escape time phenomenon [21]. Sufficient conditions for the semiglobal convergence towards the ‘second-order sliding set’ $\sigma = \dot{\sigma} = 0$ are given via appropriate Lyapunov analysis. Despite the complex nonlinear and multivariable nature of the control problem under investigation, the controller is simple to tune and to implement. The properties of the presented scheme (especially the assumed growth conditions on the uncertain terms) appear also to generalize the available results in the SISO case (see, e.g., [22, 23]).

The structure of the paper is as follows. In Section 2, the problem statement and the underlying assumptions are presented and commented. In Section 3, the proposed controller is described, and its convergence features are demonstrated by Lyapunov analysis. Some implementation aspects are discussed as well within dedicated remarks. Section 4 deals with the simulation results, and Section 5 presents some concluding remarks and possible directions for future research.
1.3. Notation

- For a vector $x = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ denote
  \[ \text{sign}(x) = \text{col}([\text{sign}(x_1), \text{sign}(x_2), \ldots, \text{sign}(x_n)])^T, \]
  \[ |x| = \text{col}(|x_1|), \quad |x|^2 = \text{col}(x_i^2), \]
  \[ \|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left[ \sum_{i=1}^{n} x_i^2 \right]^{1/2}. \]

The next relationship will be used in the sequel [24]:
  \[ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2. \]

- For a state-dependent square matrix $Q(\cdot)$ of order $m$, denote by $q_{ij}(\cdot)$ ($i, j = 1, \ldots, m$) its generic elements, and by $\lambda^i_Q(\cdot)$ ($i = 1, \ldots, m$) its eigenvalues. Also denote by $Q^s(\cdot)$ its symmetrical part, that is, $Q^s(\cdot) = 1/2 [Q(\cdot) + Q^T(\cdot)]$, which fulfills the next relation for any vector $z \in \mathbb{R}^m$ [24]:
  \[ z^T Q(\cdot) z = z^T Q^s(\cdot) z. \]

- $I_m$ denotes the $m$th order identity matrix.
- The notation ‘uniform relative degree $r_d = [r, r, \ldots, r]$’ is abridged to ‘relative degree $r$’.

**Definition 1**
A continuous function $f(\cdot) : [0, a] \to [0, \infty)$ is said to be of class $K$ if it is strictly increasing and $f(0) = 0$.

**Definition 2**
A state-dependent square matrix $Q(\cdot)$ of order $m$ is said to be diagonally dominant if
  \[ |q_{ii}(\cdot)| > \sum_{j=1, j \neq i}^{m} |q_{ij}(\cdot)|, \quad i = 1, 2, \ldots, m. \]

2. PROBLEM STATEMENT

Consider a nonlinear uncertain MIMO system in block normal form
  \[ \dot{w} = h(w, \sigma, \dot{\sigma}), \quad (11) \]
  \[ \ddot{\sigma} = B^{-1}(\sigma, w) [\Psi(\sigma, \dot{\sigma}, w) + u], \quad (12) \]
where $\sigma \in \mathbb{R}^n$ is the sliding vector to nullify, $u \in \mathbb{R}^n$ is the control vector, and $w \in \mathbb{R}^m$ is the state vector of the internal dynamics. The control task is to attain asymptotically the conditions $\sigma = \dot{\sigma} = 0$ despite the uncertain multivariable dynamics of the considered system.

It is assumed that only the $\text{sign}$ of the sliding vector and of its derivative are measured. The next assumptions are made concerning the initial conditions and the uncertain matrices of system (11)–(12).

$A_1$
There are three known constants $\sigma_0, \dot{\sigma}_0$, and $w_0$ such that
  \[ \|\sigma(0)\|_1 \leq \sigma_0, \quad \|\dot{\sigma}(0)\|_1 \leq \dot{\sigma}_0, \quad \|w(0)\|_1 \leq w_0. \]
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Two class $K$ functions $\rho_M(\cdot)$ and $m_M(\cdot)$ and three positive constants $\rho_B$, $\gamma_0$, and $\gamma_1$ can be found such that the generic entries $b_{i,j}(\cdot)$ of the matrix $B(\cdot)$ and the eigenvalues $\lambda_{B^p}(\cdot)$ of its symmetric part $B^p(\cdot) = 0.5[B(\cdot) + B^T(\cdot)]$ fulfill the next restrictions:

$$\left| b_{i,j}(\sigma, w) \right| \leq \rho_B (\|\sigma\|_1 + \|w\|_1) + \rho_B, \quad \forall (\sigma, \dot{\sigma}, w) \in M, \quad \forall i, j, = 1, \ldots, m,$$

$$0 < \gamma_0 \leq \lambda_{B^p}(\sigma, w) \leq m_M (\|\sigma\|_1 + \|w\|_1) + \gamma_1,$$

where $M \subset R^n \times R^n \times R^m$ is a sufficiently large compact domain including the origin. $B(\cdot)$ is further assumed to be nonsingular in the same domain $M$, and the $1$-norm of its inverse $B^{-1}(\cdot)$ is assumed to be a bounded, and possibly unknown, function of $\|\sigma\|_1$ and $\|w\|_1$.

Two positive constants $k_0$ and $k_2$ and two class $K$ functions $k_1(\cdot)$ and $k_3(\cdot)$ can be found such that the next growth restrictions hold:

$$\|\Psi(\sigma, \dot{\sigma}, w)\|_1 \leq k_0 + k_1 (\|\sigma\|_1 + \|\dot{\sigma}\|_1 + \|w\|_1), \quad \forall (\sigma, \dot{\sigma}, w) \in M,$$

$$\|\dot{B}(\sigma, \dot{\sigma}, w)\|_1 \leq k_2 + k_3 (\|\sigma\|_1 + \|\dot{\sigma}\|_1 + \|w\|_1),$$

Two class $K$ functions $k_4(\cdot)$ and $k_5(\cdot)$ can be found such that the solution of the internal dynamics (11) fulfills the following input-to-state stability property:

$$\|w(t)\|_1 \leq k_4 (\|w(0)\|_1) + k_5 \left( \sup_{0 \leq \tau \leq t} \|\sigma(\tau)\|_1 + \sup_{0 \leq \tau \leq t} \|\dot{\sigma}(\tau)\|_1 \right).$$

Because of Assumption $A_1$, we are casting a semiglobal stabilization problem in the sense that it is fixed a priori compact set (possibly arbitrarily large) containing the admissible initial conditions $(\sigma(0), \dot{\sigma}(0), w(0))$. Assumption $A_2$ implies the main requirement that the eigenvalues of the matrix $B^p(\sigma, w)$ (the symmetric part of $B(\sigma, w)$) should be state-dependent positive-definite functions subject to very general growth restrictions. Assumption $A_4$ is a quite general input-to-state stability condition for the internal dynamics (11) (see [24, chapter 4.9]).

3. MULTIVARIABLE SECOND-ORDER SMC

In order to stabilize system (11)–(12) under Assumptions $A_1$–$A_4$, the following discontinuous control law, which is a multivariable version of the ‘twisting’ 2-SMC algorithm [13], is suggested:

$$u(t) = -a \text{sign}(\sigma) - b \text{sign}(\dot{\sigma}),$$

where the bold sign$(\cdot)$ function should be understood in accordance with (5) and $a$ and $b$ are positive constants. The following Theorem, which constitutes the main result of this paper, is proven.

Theorem 1

Consider system (11)–(12) satisfying Assumptions $A_1$–$A_4$. Then, for any compact region of the initial conditions, as specified in Assumption $A_1$, there exists a constant $\overline{M}$ such that controller (19), with its constant parameters chosen in accordance with the inequalities

$$b > \overline{M}, \quad a > b + \overline{M},$$

ensures the asymptotic convergence of the system trajectories to the manifold $\sigma = \dot{\sigma} = 0.$
Proof of Theorem 1

The proof is broken in two separate steps.

1. Boundedness of the state trajectories

Consider the Lyapunov function

$$V(t) = \frac{1}{2} \dot{\sigma}^T B(\sigma) \dot{\sigma} + a \|\sigma\|_1,$$  \hspace{1cm} (21)

which, by employing the relation (9) and according to Assumption $A_2$, turns out to be a positive-definite function. Its time derivative along the trajectories of the uncertain system (12), (19) is

$$\dot{V}(t) = \dot{\sigma}^T \left[ \Psi + \frac{1}{2} \dot{B} \dot{\sigma} + u + a \dot{\sigma}^T \text{sign}(\sigma) \right] + a \dot{\sigma}^T \text{sign}(\sigma) = \dot{\sigma}^T \left[ \Psi + \frac{1}{2} \dot{B} \dot{\sigma} - a \text{sign}(\dot{\sigma}) - b \text{sign}(\dot{\sigma}) \right]$$  \hspace{1cm} (22)

Let us define constant $R_0$ as the largest possible initial value of $V(t)$ at $t = 0$. By considering (21) and (13), (15), along with well-known properties of the quadratic forms [24], yields

$$V(0) \leq R_0 = \frac{1}{2} \left[ m_B (\sigma_0 + w_0) + \gamma_1 \right] \dot{\sigma}_0^2 + a \sigma_0.$$  \hspace{1cm} (23)

Now consider an arbitrary constant $R$ such that $R > R_0$. Clearly, by considering (23), the next condition holds in the time interval $[0, \tau]$ for sufficiently small $\tau$

$$V(t) \leq R, \quad t \in [0, \tau].$$  \hspace{1cm} (24)

The next Lemma 1, which will be useful for the next developments, is proven.

Lemma 1

Consider the Lyapunov function (21) and let $R > 0$ be an arbitrary constant. Then, the inequality $V \leq R$ implies the following relations:

$$\|\sigma\|_1 \leq \frac{R}{a}. \hspace{1cm} (25)$$

$$\|\dot{\sigma}\|_1 \leq \sqrt{2nR \frac{\gamma_0}{\gamma_0}}. \hspace{1cm} (26)$$

$$\dot{\sigma}^T B \dot{\sigma} \leq [m_B (\|\sigma\|_1 + \|w\|_1) + \gamma_1] \sqrt{2R \frac{\gamma_0}{\gamma_0}} \|\dot{\sigma}\|_1. \hspace{1cm} (27)$$

$$\sigma^T B \dot{\sigma} \geq -\frac{n}{2} [\rho_B (\|\sigma\|_1 + \|w\|_1) + \rho_{B_0}] \left[ \frac{R}{a} \|\sigma\|_1 + \frac{1}{\gamma_0} \dot{\sigma}^T B \dot{\sigma} \right]. \hspace{1cm} (28)$$

Proof of Lemma 1

The four conditions $C_1$–$C_4$ are demonstrated in the listed order. Condition $C_1$ is trivially derived by the following chain of inequalities:

$$\dot{V} = \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} + a \|\sigma\|_1 \leq R \Rightarrow a \|\sigma\|_1 \leq R \Rightarrow \|\sigma\|_1 \leq R/a. \hspace{1cm} (29)$$

To prove Condition $C_2$, let us begin with the next implication:

$$\dot{V} = \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} + a \|\sigma\|_1 \leq R \Rightarrow \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} \leq R. \hspace{1cm} (30)$$

In light of (9) and (15), and by exploiting the standard properties of the quadratic forms (see, e.g., [24]), the next relationship holds:
\[ \frac{1}{2} \gamma_0 \| \dot{\sigma} \|^2 \leq \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} \leq \frac{1}{2} [ m_B (\| \sigma \|_1 + \| w \|_1) + \gamma_1 ] \| \dot{\sigma} \|_2^2. \] (31)

The second inequality in (30), once considered together with (31), implies that
\[ \| \dot{\sigma} \|_2^2 \leq \frac{2R}{\gamma_0} \Rightarrow \| \dot{\sigma} \|_2 \leq \sqrt{\frac{2R}{\gamma_0}}. \] (32)

From the second inequality in (32), and by taking into account (8), Condition \( C_2 \) directly follows.

To prove Condition \( C_3 \), it is sufficient to combine (31) and (32), which leads to the following chain of inequalities (condition (8) is used in the last derivation):
\[ \dot{\sigma}^T B \dot{\sigma} \leq [ m_B (\| \sigma \|_1 + \| w \|_1) + \gamma_1 ] \| \dot{\sigma} \|_2^2 \leq [ m_B (\| \sigma \|_1 + \| w \|_1) + \gamma_1 ] \sqrt{\frac{2R}{\gamma_0}} \| \dot{\sigma} \|_1. \] (33)

To prove Condition \( C_4 \), let us begin with the next simple relationship
\[ \sigma^T B \dot{\sigma} = \sum_{i,j=1,...,n} b_{ij} \sigma_i \dot{\sigma}_j, \] (34)

with \( \sigma_i \) and \( \dot{\sigma}_j \) being the \( i \)th and \( j \)th entry of \( \sigma \) and \( \dot{\sigma} \), respectively. A well-known trivial inequality establishes that, given any two real numbers \( x \) and \( y \), it holds that \( xy \geq -1/2(x^2 + y^2) \). On the basis of such inequality, and taking into account (15) and (8), it follows that
\[ \sigma^T B \dot{\sigma} \geq -\frac{1}{2} \sum_{i,j=1,...,n} b_{ij} (\sigma_i^2 + \dot{\sigma}_j^2) \geq -[\rho_B (\| \sigma \|_1 + \| w \|_1) + \rho_B_0] \sum_{i,j=1,...,n} (\sigma_i^2 + \dot{\sigma}_j^2) = \] \[ = -\frac{n}{2} [\rho_B (\| \sigma \|_1 + \| w \|_1) + \rho_B_0] [\| \sigma \|_2^2 + \| \dot{\sigma} \|_2^2]. \] (35)

By (25) and (8), we obtain
\[ \| \sigma \|_2 \leq \| \sigma \|_1 \leq \frac{R}{a} \| \sigma \|_1, \] (36)

and the first inequality in (31) yields
\[ \| \dot{\sigma} \|_2 \leq \frac{1}{\gamma_0} \| \dot{\sigma} \|_2 \| B \dot{\sigma} \|_1. \] (37)

Now considering (36) and (37) into (35), the claimed fourth condition (28) follows directly. Lemma 1 is proven.

By virtue of Lemma 1, and on the basis of (24), in the time interval \( t \in [0, \tau] \), explicit constant bounds for the norm of the elements \( \Psi \) and \( 0.5B \dot{\sigma} \) appearing in (22) can be derived. To this end, consider (18) together with the bounds (13), and (25) and (26). It yields that a positive constant \( \bar{w} \) can be found such that
\[ \| w(t) \|_1 \leq \bar{w} = \kappa_4 (w_0) + \kappa_5 \left( \frac{R}{a} + \sqrt{\frac{2\eta R}{\gamma_0}} \right), \quad t \in [0, \tau]. \] (38)

\[^1\text{It suffices to consider that } (x + y)^2 = x^2 + y^2 + 2xy \geq 0, \text{ and, after reordering, to derive the claimed inequality directly.}\]
By considering the same bounds into (16) and (17), it follows that two positive constants $\Psi$ and $\Omega_d$ can be found such that

$$
\|\Psi(t)\|_1 \leq \Psi = \kappa_0 + \kappa_1 \left(\frac{R}{a} + \sqrt{\frac{2nR}{\gamma_0} + \overline{w}}\right), \quad t \in [0, \tau],
$$

(39)

$$
\|\Omega(t)\|_1 \leq \Omega_d = \kappa_2 + \kappa_3 \left(\frac{R}{a} + \sqrt{\frac{2nR}{\gamma_0} + \overline{w}}\right), \quad t \in [0, \tau].
$$

(40)

Then, by (39) and (40) along with Condition $C_2$ of Lemma 1, constant $\Gamma$ can be found such that

$$
\|\Psi + \frac{1}{2} \Omega \|_1 \leq \|\Psi\|_1 + \|\Omega\|_1 \leq \Gamma = \Psi + \Psi \sqrt{\frac{2nR}{\gamma_0} + \overline{w}}, \quad t \in [0, \tau].
$$

(41)

Considering (23) and (38)–(40) into (41), and making simple manipulations, one can derive the following explicit form for the constant $\Gamma$:

$$
\Gamma = \kappa_0 + \kappa_1 (\varphi_0) + (k_2 + k_3 (\varphi_0)) \sqrt{\frac{2nR}{\gamma_0}},
$$

(42)

$$
\varphi_0 = \frac{R}{a} + \sqrt{\frac{2nR}{\gamma_0} + \kappa_4 (w_0) + \kappa_5 \left(\frac{R}{a} + \sqrt{\frac{2nR}{\gamma_0}}\right)},
$$

(43)

$$
R = \eta R_0 = \eta \left[\frac{1}{2} [m_{\varphi} (\sigma_0 + w_0) + \gamma_1] \hat{\sigma}_0^2 + a \sigma_0 \right], \quad \eta > 1,
$$

(44)

which depends on the constants and functions given in Assumptions $A_1$–$A_4$ and on the arbitrary constant $\eta > 1$. Equation (44) follows from the requirement that $R$ should be larger than $R_0$.

On the basis of (41), equation (22) can be further manipulated as follows:

$$
\hat{V}(t) = \hat{\sigma}^T [\Psi + \frac{1}{2} \Omega \hat{\sigma}] - b \|\hat{\sigma}\|_1 \leq -(b - \Gamma) \|\hat{\sigma}\|_1, \quad t \in [0, \tau].
$$

(45)

The controller parameter $b$ is tuned according to the inequality (20), $b > \overline{M}$, then let us choose constant $\overline{M}$ according to

$$
\overline{M} > \Gamma,
$$

(46)

which implies, by construction, that $b > \Gamma$ and that the next estimation

$$
\hat{V}(t) \leq -(b - \Gamma) \|\hat{\sigma}\|_1, \quad b > \Gamma, \quad t \in [0, \tau],
$$

(47)

holds, thus guaranteeing that the positive-definite function $\hat{V}$ does not increase in the time interval $t \in [0, \tau]$, that is, that

$$
V(t_2) \leq V(t_1), \quad \tau \geq t_2 \geq t_1 \geq 0.
$$

(48)

Then, $V(\tau) \leq V(0) \leq R_0$, and, by simple iteration of the same procedure, the inequality $V(t_k) < V(t_{k+1})$ can be derived, where $t_k$ and $t_{k+1}$ are arbitrary time instants such that $(k + 1)\tau \geq t_{k+1} \geq t_k \geq k \tau$ and $k = 0, 1, 2, \ldots$. It means that parameter $\tau$ can be extended to infinity within the relations (47) and (48), and, in turn, this implies that

$$
\hat{V}(t) \leq -(b - \Gamma) \|\hat{\sigma}\|_1, \quad t \in [0, \infty),
$$

(49)

and that there exists $R > 0$ such that

$$
V(t) \leq R, \quad \forall t \in [0, \infty).
$$

(50)

Hence, Conditions $C_1$–$C_4$ given in Lemma 1 hold at any time $t \geq 0$ if one considers the value of $R$ given in (44). Therefore, it has been shown that the uncertain system (11)–(12), (19)–(20) has
bounded trajectories in the semi-infinite time interval \( t \in [0, \infty) \); furthermore, a bounded invariant set for the \((\sigma, \dot{\sigma})\) trajectories in the form of an \( R \)-domain can be computed explicitly:

\[
D_R = \{(\sigma(t), \dot{\sigma}(t)) : V(\sigma(t), \dot{\sigma}(t)) \leq R\}.
\]  

Additionally, the nonincreasing evolution of the Lyapunov function \( \dot{V}(t) \) guarantees that any trajectory starting from the \( \epsilon \)-domain \( D_\epsilon \), with arbitrarily small \( \epsilon < R \) (possibly tending to zero), will not leave such a domain at any subsequent time. Hence, this demonstrates the local stability (yet not asymptotical) of the system motion in a bounded vicinity of the second-order sliding manifold \((\sigma, \dot{\sigma}) = (0,0)\).

2. Semiglobal asymptotic stability

In order to demonstrate the asymptotical convergence of the system trajectories towards the second-order sliding manifold \( \sigma = \dot{\sigma} = 0 \), we now construct a parameterized family of local Lyapunov functions \( V_R(t) \), \( R > 0 \), such that each \( V_R(t) \) is well defined on the corresponding compact set (51). More precisely, we look for a modified Lyapunov function \( V_R \), which is positive definite in \( D_R \setminus \{(0,0)\} \) and zero at the origin, whereas its time derivative, computed along the trajectories of the uncertain system (11)–(12), (19), with initial conditions fulfilling Assumption \( A_1 \), is to be negative definite for all \((\sigma, \dot{\sigma})\) belonging to the invariant domain \( D_R \setminus \{(0,0)\} \).

A parameterized family of Lyapunov functions with the properties mentioned can be constructed as follows, by ‘augmenting’ \( \dot{V}(t) \) with the sign-indefinite term \( U(t) = k_R\sigma^T B(\cdot) \dot{\sigma} \), where \( k_R \) is an appropriate constant. Thus, define

\[
V_R(t) = \dot{V}(t) + U(t) = \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} + a \|\sigma\|_1 + k_R\sigma^T B \dot{\sigma}.
\]  

By Condition \( C_4 \) of Lemma 1, the Lyapunov function (52) can be estimated as

\[
V_R(t) \geq \frac{1}{2} \dot{\sigma}^T B \dot{\sigma} + a \|\sigma\|_1 - k_R \frac{R}{2} [\rho_B(\|\sigma\|_1 + \|w\|_1) + \rho_B] \left[ \frac{R}{a} \|\sigma\|_1 + \frac{1}{\gamma} \|\sigma\|_1 \right] = \left( \frac{1}{2} - \frac{n k_R [\rho_B(\|\sigma\|_1 + \|w\|_1) + \rho_B]}{2\gamma} \right) \sigma^T B \dot{\sigma} + \left( a - \frac{n k_R [\rho_B(\|\sigma\|_1 + \|w\|_1) + \rho_B]}{2a} \right) \|\sigma\|_1.
\]  

We now take advantage from the fact that in the first part of the proof, it has been shown that the system trajectories belong to the invariant compact set \( D_R \) (51) starting from the initial time instant on, in accordance with (50). Then, Conditions \( C_1 \)–\( C_4 \) given in Lemma 1, as well as the inequality (38), hold true at any time \( t \geq 0 \). Considering (25) and (38), it yields that a constant \( \rho_B^* \) can be found such that

\[
\rho_B(\|\sigma\|_1 + \|w\|_1) \leq \rho_B^* = \rho_B \left( \frac{R}{a} + \kappa_4 (w_0) + \kappa_5 \left( \frac{R}{a} + \frac{2nR}{\gamma} \right) \right).
\]  

Let the weight parameter \( k_R \) be small enough, in accordance with the following restriction:

\[
k_R < \min \left\{ \frac{2}{n(\rho_B^* + \rho_B)}, \frac{2a^2}{n(\rho_B^* + \rho_B)R} \right\}.
\]  

By considering (53) and (54), it turns out that (55) guarantees that the family of functions \( V_R \) so constructed is positive definite in the domain \( D_R \setminus \{(0,0)\} \). Now compute the derivative \( \dot{V}_R(t) = \dot{V}(t) + U(t) \) along the trajectories of the uncertain system (11)–(12), (19). It yields

\[
\dot{U}(t) = k_R\sigma^T B \dot{\sigma} + k_R\sigma^T z,
\]  

where vector \( z \) is given by

\[
z = \Psi + \dot{B} \dot{\sigma} - a \text{sign} \sigma - b \text{sign} \dot{\sigma}.
\]  

According to the controller tuning formulas (20) and (46), one has that

\[
a > b + M > b + \Gamma > b + \|\Psi + \dot{B} \dot{\sigma}\|_1,
\]
then it is straightforward to derive the next condition:
\begin{equation}
\sigma^T z \leqslant -(a - b - \Gamma)\|\sigma\|_1.
\end{equation}

Considering (49) and the relations (56)–(59), it follows that
\begin{equation}
\dot{V}_R \leqslant -(b - \Gamma)\|\dot{\sigma}\|_1 + k_R \sigma^T B \dot{\sigma} - (a - b - \Gamma)\|\sigma\|_1.
\end{equation}

By virtue of Condition $C_3$ of Lemma 1, it is possible to further manipulate (60) as
\begin{equation}
\dot{V}_R \leqslant - \left[ b - \Gamma - k_R [m_B (\|\sigma\|_1 + \|w\|_1) + \gamma_1] \sqrt{\frac{2R}{\gamma_0}} \right] \|\dot{\sigma}\|_1 - [a - b - \Gamma]\|\sigma\|_1. \tag{61}
\end{equation}

Considering (25) and (38), it yields that a constant $m_B^*$ can be found such that, at any $t > 0$,
\begin{equation}
m_B (\|\sigma\|_1 + \|w\|_1) \leqslant m_B^* = m_B \left( \frac{R}{a} + \kappa_4 (v_0) + \kappa_5 \left( \frac{R}{a} + \sqrt{\frac{2nR}{\gamma_0}} \right) \right). \tag{62}
\end{equation}

Thus, by taking into account the next inequalities involving the arbitrary coefficient $k_R$ (more restrictive than (55)),
\begin{equation}
k_R < \min \left\{ \frac{\lambda_B}{n(\rho_B^2 + \rho_b)}, \frac{2a^2}{n\rho_B^3 R}, \frac{(b - \Gamma)}{m_B^* + \gamma_1} \sqrt{\frac{\gamma_0}{2R}} \right\}, \tag{63}
\end{equation}
it is concluded that $\dot{V}_R$ is negative definite for all $(\sigma, \dot{\sigma}) \in D_R \setminus \{(0, 0)\}$. To complete the proof, it remains to demonstrate that $\sigma(t)$ and $\dot{\sigma}(t)$ tend to zero as $t \to \infty$. For this purpose, let us integrate the relation
\begin{equation}
\dot{V}_R(t) \leqslant -(a - b - \Gamma)\|\sigma\|_1, \tag{64}
\end{equation}
straightforwardly resulting from the negative definiteness of all terms in the right-hand side of (61), to conclude that
\begin{equation}
\int_0^\infty \|\sigma(t)\|_1 \, dt < \infty. \tag{65}
\end{equation}

According to (26), the integrand $\omega_1(t) = \|\sigma(t)\|_1$ of (65) possesses a time derivative
\begin{equation}
\dot{\omega}_1(t) = \dot{\sigma}^T(t) \text{ sign } \sigma(t) \tag{66}
\end{equation}

such that
\begin{equation}
\|\dot{\omega}_1(t)\| \leqslant \|\dot{\sigma}(t)\|_1 \leqslant \sqrt{\frac{2nR}{\gamma_0}} \tag{67}
\end{equation}
on the semi-infinite time interval $t \in [0, \infty)$. By relying on the uniform boundedness property (67), the asymptotic convergence of $\sigma(t)$ to zero is then verified by applying the Barbalat lemma (see [25, chapter 4.5]). To prove the same property for $\dot{\sigma}(t)$, the next relation is derived from (61) and (62),
\begin{equation}
\dot{V}_R(t) \leqslant - \left[ b - \Gamma - k_R [m_B^* + \gamma_1] \sqrt{\frac{2R}{\gamma_0}} \right] \|\dot{\sigma}\|_1 = -\theta \|\dot{\sigma}\|_1, \tag{68}
\end{equation}
where $\theta > 0$ according to (63), which allows to conclude that
\begin{equation}
\int_0^\infty \|\dot{\sigma}(t)\|_1 \, dt < \infty. \tag{69}
\end{equation}

The integrand $\omega_2(t) = \|\dot{\sigma}(t)\|_1$ of (69) possesses a time derivative
\begin{equation}
\dot{\omega}_2(t) = \dot{\sigma}^T(t) \text{ sign } \dot{\sigma}(t) \tag{70}
\end{equation}
such that
\[ |\dot{\sigma}_2(t)| \leq \| \dot{\hat{\sigma}}(t) \|_1. \] (71)

To assess the boundedness of \( \| \dot{\hat{\sigma}}(t) \|_1 \), let us refer to the sliding vector dynamics (12). It has been shown that \( \| \sigma(t) \|_1 \) and \( \| \dot{\sigma}(t) \|_1 \) are uniformly bounded by appropriate constants on the semi-infinite time interval \( t \in [0, \infty) \). By (18), the same property holds for \( \| w(t) \|_1 \). The norm of the control vector \( u(t) \) is also uniformly bounded by construction, as it readily follows from (19). Hence, considering Assumptions A1–A4, \( \| \dot{\hat{\sigma}}(t) \|_1 \) will be uniformly bounded on the semi-infinite time interval \( t \in [0, \infty) \), too, and all the previous conditions imply, by Barbalat Lemma [25, chapter 4.5], that \( \| \dot{\hat{\sigma}}(t) \|_1 \to 0 \) as \( t \) grows to infinity. It is therefore proven that the origin of the \( (\sigma, \dot{\sigma}) \) plane is asymptotically attracting the \( (\sigma, \dot{\sigma}) \) trajectories of the uncertain system (11)–(12) with the discontinuous feedback (19)–(20). Theorem 1 is proven. \( \Box \)

**Remark 1**
The controller tuning conditions (20) involve the evaluation of the constant \( \overline{M} \) that will affect the magnitude of the discontinuous control terms. According to the stability analysis, \( \overline{M} \) should be chosen according to the inequality (46), with the \( \Gamma \) constant defined in (42)–(44). As usual in the SMC implementation, the tuning conditions derived for the discontinuous control magnitudes are conservative, and, whenever exactly applied, would give rise to unnecessarily large values of the discontinuous control effort. It is more appropriate and effective, in practice, to tune the controller by starting from a reasonably small initial guess for \( \overline{M} \) and progressively increasing it until satisfactory system behavior is achieved.

**Remark 2**
The designed control vector \( u(t) \) is discontinuous, which might seem to prevent the application of the suggested approach in, for example, mechanical systems with forces or torques as input variables. In some cases, however, the input vector \( u(t) \) in a control problem plays the role of a fictitious control variable, representative of the first time derivative of the ‘physical’ control input (e.g., a force or a torque). In this regard, the discontinuity of \( u(t) \) is not an issue, and a continuous force or torque will be produced by the given control scheme after integrating the discontinuous control derivative output by the relevant controller. This approach has been extensively used in many practical applications of 2-SMC to SISO or MIMO engineering systems (among them, robotic manipulators, pantographs, and overhead cranes, see [19]). A different 2-SMC control design could be implemented if one wants to obtain a continuous control vector \( u(t) \) for a relative degree two system. One can define a new sliding variable vector \( S(t) = \hat{\sigma}(t) + c \sigma(t) \), with \( c > 0 \), and then implement the continuous control algorithm \( u(t) = - \int_0^t \left[ a(\text{sign}(S(t)) + b \text{sign}(S(t))) \right] \mathrm{d}t. \) The convergence of the proposed scheme could be proven using almost analogous arguments to those used in the present paper. It is worthy to stress that the control law, thus modified, would require the measurement of \( \hat{\sigma}(t) \), which might not be easy to obtain in practice.

**4. SIMULATIONS**

The proposed scheme is simulated by considering a randomly generated 10th order nonlinear system of the type (11)–(12), with the sliding vector and control input of dimension 8 \( (\sigma, u \in \mathbb{R}^8) \) and the internal dynamics of dimension 2 \( (w \in \mathbb{R}^2) \). Particularly, the following dynamics are considered:
\[
\begin{align*}
\dot{w} &= A_w w + F_1 \cdot (|\sigma|^2 + |\dot{\sigma}|^2), \\
\dot{\sigma} &= B^{-1} \left[ F_2 \cdot (|\sigma|^2 + |\dot{\sigma}|^2) + G |w| + (2 + \sin(\sigma))u \right],
\end{align*}
\] (72) (73)

where \( A_w, F_1, B, F_2, G \) are constant matrices of appropriate dimension, with \( A_w \) being a Hurwitz matrix. The meaning of the vector terms \( |\sigma|^2 \) and \( |\dot{\sigma}|^2 \) was explained in Section 1.3.

The considered nonlinear dynamics is open loop unstable and subject to the finite escape time phenomenon. It is easy to show that Assumptions A2–A4 are actually fulfilled and the involved
functions exist. Assumption $A_5$ implies that matrix $A_w$ must be Hurwitz. All matrices have been randomly generated with the MATLAB (The MathWorks, Inc., Natick, MA, USA) ‘rand’ function by taking into account the constraints on $B$ and $A_w$. The latter matrices are chosen as follows:

$$A_w = \begin{bmatrix} -0.7578 & -0.7411 \\ -0.7411 & -0.8381 \end{bmatrix}.$$  \hfill (74)

To guarantee that $B$ has positive eigenvalues, the formula $B = X_1X_1^T$ has been used, with $X_1$ being an 8×8 random matrix, whereas $F_1$, $F_2$, and $G$ are directly generated as random matrices of appropriate dimension. All the generated random matrix entries are uniformly distributed in the interval $\{-1, 1\}$.

The system initial conditions are taken as $\sigma_i(0) = \dot{\sigma}_i(0) = 0.5, (i = 1, \ldots, 8)$ and $w(0) = [1, 1]$. The simulations are performed in the Simulink (The MathWorks, Inc.) environment using the fixed-step Euler ODE solver with the step size $T_s = 10^{-6}$ s. In a first experiment, the input vector is set to zero to investigate the finite escape time of the system trajectories. Figure 1 reports the sliding vector components that escape to infinity in less than 0.12 s.

In TEST 2, the proposed controller is implemented, with the constant $\overline{M}$ in the tuning formula (20) set to $\overline{M} = 70$ and the controller parameters selected accordingly as $b = 80$ and $a = 160$. Figure 2 (left) reports the sliding variable vector components, and Figure 2 (right) shows the first discontinuous control component $u_1$. Figure 3 (left) reports the corresponding components of the sliding variable vector derivative.

In the successive TEST 2, another instance of the random model matrices has been generated, and the same $a, b$ controller parameters have been used. Figure 3 (right) reports the sliding variable components in TEST 2, which confirms the robust performance of the proposed controller. In order to check the sample-and-hold effects, the controller has been implemented digitally with measurement step interval of $T_m = 10^{-4}$ s (TEST 3) and $T_m = 10^{-5}$ s (TEST 4). The sliding variable 2-norm $\|\sigma\|_2$ is analyzed in the steady state as a measure of the overall sliding motion accuracy.

![Figure 1: Components of $\sigma$ with zero input.](image)
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Figure 2. Components of $\sigma$ (left plot) and first control component $u_1$ (right plot) in TEST 1.

Figure 3. Components of $\dot{\sigma}$ in TEST 1 (left plot) and components of $\sigma$ in TEST 2 (right plot).

Figure 4. The sliding variable 2-norm in TEST 3 (left) and TEST 4 (right).

Figure 4, which compares the zoomed plots of the sliding variable 2-norm in the steady state in TEST 3 and TEST 4, shows that the accuracy of the sliding motion in TEST 4 is almost 100 times higher, which means that the order of accuracy of the sliding motion with respect to the sampling time is $O(T_m^2)$. 

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5. CONCLUSIONS

A multi-input version of the ‘twisting’ 2-SMC algorithm has been suggested in order to control a class of nonlinear uncertain dynamics of vector relative degree [2, 2, . . . , 2]. The main improvements over the existing literature on multi-input 2-SMC are the following: (i) the assumptions on the HFG matrices are relaxed and (ii) the boundedness assumptions on the drift vector field appearing in the sliding variable dynamics are generalized. Interesting lines of activity that call for future investigation would be the analysis of the finite-time convergence properties of the algorithm to deal with sliding variable dynamics having higher relative degrees and to dispense with the assumption of positive definiteness for the HFG matrix, thereby generalizing the approach to systems with uncertain control direction.

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