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On the second-order sliding mode control of nonlinear systems with uncertain control direction

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A B S T R A C T

This note deals with the implementation of a second-order sliding mode control algorithm for a class of nonlinear systems in which the sign of the high-frequency gain, though constant, is unknown. A specific second-order sliding mode control algorithm, the “Suboptimal” algorithm, is properly modified in order to face the uncertainty in the control direction. It is shown that after a finite time the uncertain sign is identified and the standard finite time convergence takes place from that time on. Simulation results are provided.

1. Introduction

In this note we consider a class of systems affected by bounded matching uncertainties where the sliding output has relative degree two and whose main pathology consists in having a constant uncertain control direction. This work can be then considered as an ideal continuation and generalization of the previous paper (Bartolini, Ferrara, & Giacomini, 2003) in which nonlinear uncertain systems having unknown constant high-frequency gain (HFG) sign and known relative degree 1 were considered.

The main ingredient of the proposed scheme is the introduction of an adaptive gain, switching between the values +1 and −1, which premultiplies the control variable. It is shown that after a finite number of switchings the adaptive gain settles to the constant, “correct”, value guaranteeing the finite time convergence of the system to the origin.

We consider systems in normal form as follows

\[
\dot{x} = f(x, s, \dot{s}) \quad (1)
\]

\[
\dot{s} = \varphi(x, s, \dot{s}, t) + g^* \gamma(x, s, \dot{s}, t)u(t), \quad (2)
\]

where \(s \in \mathbb{R}\) represents the sliding variable, \(u \in \mathbb{R}\) is the control input, \(x \in \mathbb{R}^n\) is the internal dynamics vector. Let the internal dynamics (1) be input-to-state (ISS) stable (Khalil, 2002), and let the scalar uncertain drift and gain functions fulfill the following inequalities for some constants \(F, G_1\) and \(G_2\).

\[
|\varphi(\cdot)| \leq F, \quad 0 < G_1 \leq \gamma(\cdot) \leq G_2. \quad (3)
\]

The uncertain constant control gain \(g^* \in [-1, +1]\) represents the uncertainty on the HFG sign. The solutions of (1)–(2) will be understood in the Filippov sense (Filippov, 2000) in order to allow the use of discontinuous control. The above conditions imply that the control gain \(\gamma(\cdot)\) is assumed to have known sign.

In the adaptive control literature the case of systems with unknown HFG sign has been treated by means of the so-called Nussbaum functions (Nussbaum, 1983) which have been applied to deal with large classes of linear and nonlinear processes with relative degree one, stable zero-dynamics, and unknown control direction (Ryan, 1995). This method does not guarantee exponential convergence, and, furthermore, may frequently lead to the presence of peaking phenomena (Bartolini et al., 2003; Sussmann & Kokotovic, 1991).

The generalization of the adaptive control approach to systems with relative degree two and uncertain sign of the control gain has not been as much fruitful. The paper Ryan (1995), the seminal contribution to this topic, considered the presence of a persistent viscous dissipation. More recently, in the framework of the so-called VS-MRAC (Variable Structure Model Reference Adaptive Control) approach (Oliveira, Peixoto, Nunes, & Hsu, 2007) a method has been presented, based on the “monitoring function” concept, for systems having a known arbitrary relative degree by using derivative estimators based on higher-order sliding modes (see Levant (2003)) to reduce it to one. The problem was also studied in Drakunov (1993) for relative degree one sliding mode control systems.
In this paper we provide a solution based on the second-order sliding mode approach. In Section 2 the paper’s main result is presented, and a simple academic simulation example is discussed. Section 3 shows some simulation result on a fourth-order plant which is open-loop unstable, and the Section 4 draws some concluding remarks.

2. Main result

Consider system (1)-(2) and let the sliding output variable $s$ be the only signal available for measurements. Let known constants $F$, $G_1$, $G_2$ exist such that conditions (3) are fulfilled. The requirement of a constant value for the upper bound $F$ might seem to imply a local, a priori bounded, domain of attraction. As shown in Bartolini, Ferrara, Pisano, and Usai (2001), by considering “sufficiently large” value of $F$ the convergence domain can be enlarged arbitrarily (semi-global stability). The problem of making the monitoring function approach guaranteeing the convergence. The monitoring function approach in Oliveira et al. (2007) shares a similar functioning principle.

By (8), at the initial time $t = t_0$ the system belongs to the domain $D_0$. The constant $\rho(t)$ is initialized to 1, and its sign is reversed every time that the actual $D_i$ domain ($i = 0, 1, 2, \ldots$) is left for the first time. An analytical representation of this adaptation mechanism is as follows.

\textbf{Initialization:} \( \rho(0) = 1; i = 1; \)

\textbf{step 1.} if \( |s(t)| \geq \Delta_i \) then set \( \rho(t) := -1, \)
\( i := i + 1, \) and go to step 2.

\textbf{step 2.} if \( |s(t)| > \Delta_i \) then set \( \rho(t) := 1, \)
\( i := i + 1, \) and go to step 1.

We shall prove that after a finite number of switchings of $\rho(t)$ (possibly zero) the origin of the state plane is reached in finite time.

The following academic example highlights the importance of the progressive enlargement of the thickness of the domains $D_i$. Let us consider the simple perturbed double integrator $\dot{y} = \varphi + u$, with initial conditions $y(0) = 0.5, \dot{y}(0) = 1$ and uncertain constant drift term $\varphi = 1$. The controller ignores the sign of the control gain. The algorithm (6), (10), has been implemented with the parameters $U_M = 4$ and $\beta = 0.5$. Preliminarly (TEST 1), a modified definition of the regions $D_i$ is considered, namely the coefficients $\Delta_i$ are chosen such that $\Delta_i = 1, \Delta_i = \Delta_{i-1} + 1 (i = 2, 3, \ldots)$. This corresponds to the regions $D_i$ (7) having constant unitary thickness 1, i.e. $\Delta_1 = 1, \Delta_2 = 2, \Delta_3 = 3, \Delta_4 = 4, \ldots$. This means that the fundamental tuning condition (11), that will be derived later, is not fulfilled in the TEST 1. It is seen in Fig. 1-left that the system trajectory diverges. The proposed formula (8) for the computation of the sequence $\Delta_i$ has been considered in TEST 2, with the parameters $k^2 = 0.5$ and $\mu = 1$. This leads to $\Delta_1 = 1, \Delta_2 = 2, \Delta_3 = 3, \Delta_4 = 4, \ldots$. It is seen in the Fig. 1-right that the system trajectory now exhibits the desired convergence properties. The correct value of the gain $\rho(t)$ is identified after three switches, and region $D_4$ is not entered anymore.
2.1. The proposed algorithm

We are now in position of proving the following:

**Theorem 1.** Given system (1)-(3), the control algorithm (6), (10) guarantees the finite time attainment of condition $s = \hat{s} = 0$ provided that the sequence $\Delta_i$ is defined according to (8) with

$$\mu > \frac{(G_G - G_I)U_M + 2F}{G_I U_M - F}. \quad (11)$$

**Proof of Theorem 1.** The sequence of increments and decrements of $\hat{s}$ characterizes the proposed strategy. Consider the system behaviour after the domain $\mathcal{D}_t$ has been entered at the time $t = t_i$ ($i \geq 1$). At the time $t = t_i$, $|s(t_i)| = \Delta_i$. Denote $|s(t_i)| = \hat{s}_i$. Without loss of generality consider the case when both $\bar{s}$ and $\hat{s}$ are increasing in the domain $\mathcal{D}_i$ (see Fig. 2). Thus, in finite time $t = t_{i+1}$ the domain $\mathcal{D}_{i+1}$ is entered. At that time the control gain $\rho(t)$ reverses its sign, and, from this point on, $\hat{s}$ decreases in modulus. By construction, $\hat{s}_{i+1} > \hat{s}_i$ and $\hat{s}_{i+2} < \hat{s}_{i+1}$. It must be proven that, if the domain $\mathcal{D}_{i+2}$ is entered the corresponding value of $\hat{s}_{i+2}$ is strictly less than $\hat{s}_i$.

Due to the uncertainties in the system dynamics, a worst case analysis is mandatory which selects the uncertainties that maximize the modulus of $\hat{s}_{i+2}$. Clearly, by making reference to the Fig. 2, this amounts to consider the maximal positive slope in the domain $\mathcal{D}_i$ and the less negative slope in the domain $\mathcal{D}_{i+1}$. In the domain $\mathcal{D}_i$, the trajectory having the maximal positive slope is

$$s(t) = \Delta_i + \frac{1}{2} \frac{\hat{s}_i^2}{(t - t_i)}. \quad (12)$$

The value of $\hat{s}_{i+1}$ is obtained by putting $s(t) = \Delta_{i+1}$ into (12) which yields

$$\hat{s}_{i+1}^2 = \hat{s}_i^2 + 2(G_G U_M + F)(\Delta_{i+1} - \Delta_i). \quad (13)$$

In the domain $\mathcal{D}_{i+1}$ the trajectory having the less negative slope is

$$s(t) = \Delta_{i+1} - \frac{1}{2} \frac{\hat{s}_i^2}{(t - t_{i+1})}. \quad (14)$$

The value of $\hat{s}_{i+2}$ is obtained by putting $s(t) = \Delta_{i+2}$ into (14). By (13)

$$\hat{s}_{i+2}^2 = \hat{s}_i^2 + 2(G_G U_M + F)(\Delta_{i+1} - \Delta_i) - (G_I U_M - F)(\Delta_{i+2} - \Delta_{i+1}). \quad (15)$$

To ensure the convergence it must be proven that $\hat{s}_{i+2}^2 < \hat{s}_i^2$. This means that the bracketed term in (15) must be negative, i.e.

$$(G_G U_M + F)(\Delta_{i+1} - \Delta_i) < (G_I U_M - F)(\Delta_{i+2} - \Delta_{i+1}). \quad (16)$$

Note that if the thickness of the adjacent domains $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ would be constant (e.g., if $\Delta_{i+1} - \Delta_i = \Delta_{i+2} - \Delta_{i+1} = const.$) the worst case sequence $\{\hat{s}_i\}$ diverges and no contractive behaviour can be longer guaranteed.

Convergence is ensured if

$$(\Delta_{i+2} - \Delta_{i+1}) > \frac{(G_G U_M + F)}{(G_I U_M - F)}(\Delta_{i+1} - \Delta_i). \quad (17)$$

By taking into account (9) the convergence condition (17) can be expressed as follows in terms of the $\mu$ parameter.

$$1 + \mu > \frac{G_G U_M + F}{G_I U_M - F}. \quad (18)$$

It is trivial to derive from (18) the tuning inequality (11). The maximal number of switches for the gain function $\rho(t)$ can be computed. Let $\hat{s}_i$ the value of $\hat{s}$ at the time instant at which the domain $\mathcal{D}_i$ is entered. It was shown that the bracketed term in (15) is negative. Then it can be written $\hat{s}_{i+2}^2 = \hat{s}_i^2 - \xi^2$. The switching process ends when $\hat{s}_{i+2}$ becomes negative. The maximal number of switches is then $N_{\xi} = 1 + 2\xi$, where $\xi$ is the smallest integer such that $\hat{s}_{i+2}^2 - 2\xi^2 < 0$. By increasing $\mu$ it correspondingly increases the $\xi$ constant, which implies that the maximal number of switches can be reduced by increasing $\mu$.

Once the axis $\hat{s} = 0$ is hit at the generic time instant $t_i^* > t_i$ the value of $\rho(t)$ will be such that $\text{sign}(\rho(t^*)) = \text{sign}(g^*)$, and, at the same time, the domain $\mathcal{D}_i$ will never be left due to the suboptimal algorithm contraction property which establishes that successive horizontal axis crossing point are closer and closer to the origin (Bartolini et al., 2001). Then, from the time instant $t_i^*$ on, the unknown sign of the high-frequency gain has been identified and the remaining convergence proof follows the standard arguments already used in previous papers (e.g. Bartolini et al. (2001)). This conclude the proof. □

**Remark 2.** By similar considerations as those made in Bartolini et al. (2008), it can be shown that the proposed technique can be also applied, without any modification, to stabilize systems with relative degree one and having unknown sign of the high-frequency gain.

**Remark 3.** The error in estimating the sequence $s(t_{lm})$ (see the Remark 1) causes a deterioration of the ideal performance $\hat{s} = \hat{s} = 0$ stated in the Theorem 1. Actually, the variables $s(t)$ and $\hat{s}(t)$ will eventually enter a small vicinity of zero instead of converging exactly to zero (Bartolini et al., 1997, 1998).

3. Simulations

Consider the fourth-order system

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 - x_2^3 + y_1 + y_2, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= y_1^3 + y_2 + x_1 + x_2 + g^*(3 + \sin(2\pi t))u
\end{align*}$$

(19)

with the input term $u$ premultiplied by the uncertain coefficient $g^* \in [-1, 1]$ which models the assumed uncertainty on the sign of the control high-frequency gain. The internal dynamics can be easily shown to be ISS. Note that the input output dynamics may exhibit the finite escape time phenomenon if not properly controlled. The initial conditions are $x(0) = y(0) = [1, 1]$. Assume the bound $F = 15$ for the system drift term, and set coefficients $G_G = 2, G_I = 4$. By using the tuning formulas (5), (11) the parameter values $U_M = 40, \beta = 0.6, \xi_1 = 0.1, \mu = 1$ are derived. With the selected parameters the state plane is partitioned into the strips having $\Delta_1 = 0.6, \Delta_2 = 1.2$, etc. Let us first simulate the system with negative high-frequency gain $g^* = -1$ (TEST 3). Fig. 3 shows that convergence is achieved after one switch of the coefficient $\rho(t)$, and Fig. 4-left shows the corresponding trajectory in the $y_1 - y_2$ plane. Now run the simulation test putting the positive control gain $g^* = 1$ (TEST 4). The correct value for $\rho(t)$ is identified with no switches for $\rho(t)$, and fast convergence to zero is ensured (see Fig. 4-right).
4. Conclusions

A second-order sliding mode based control algorithm has been proposed to address the finite time stabilization problem for a class of nonlinear uncertain systems with relative degree two and uncertain sign of the control gain. The simulation results confirm the expected performance.

References


