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Model Reduction of Finite-State Machines by Contraction

Alessandro Giua

***Abstract*—This note discusses an approach to the model reduction of discrete event systems represented by finite-state machines.**

A set of good reduced-order approximations of a deterministic finite-state machine M can be efficiently computed by looking at its contractions, i.e., finite-state machines constructed from M by merging two states. In some particular case, it is also possible to prove that the approximations thus constructed are infimal, in the sense that there do not exist better approximations with the same number of states.

This note also defines a merit function to choose, among a set of approximations, the best one with respect to a given observed behavior.

***Index Terms*—Discrete-event systems, finite automata, reduced-order systems.**

I. Introduction

Model reduction techniques have been used in control theory to approximate high order systems with simpler ones that still capture the behavior of the original complex systems.

In this note, we consider the same problem in the framework of *discrete event systems* [7]. In particular, a discrete event system will be modeled by a *finite-state machine* (FSM) and its behavior will be given by the language generated.

A *reduced-order approximation* of a minimal deterministic FSM M with n states is a deterministic FSM M' with $n' < n$ states such that $L(M') \supset L(M)$. Let M' be an approximation of order n' of M; we say that M' is *infimal* if there does not exist another approximation M'' of order $n'' \leq n'$ such that $L(M') \supset L(M'') \supset L(M)$.

Computing infimal approximations is a complex task. The paper shows how a set of good—but possibly not infimal—approximations of a given minimal deterministic FSM M can be computed efficiently by looking at *contractions* of M, i.e., FSMs constructed from M by merging two states. In some particular case, it can also be proven that all approximations in the set thus constructed are infimal.

This note also discusses how, given a set of approximations of an FSM M, it is possible to define a merit function to choose the best approximation with respect to a given observed behavior $L_o \subseteq L(M)$.

The two requirements of having a small-order model and a tight language approximation are conflicting. The procedure presented in this paper can be recursively applied, starting with a given FSM and computing contractions until a satisfactory tradeoff between order of the model and degree of language approximation is reached.

The proposed approach is particularly useful in the case of systems composed of interconnected subsystems. It is well known that composing the FSM modules that describe the different subsystems—e.g., using the concurrent composition operator [7]—the number of states of the resulting overall model grows exponentially. The reduction of even a few states in each FSM module may lead to a significant simplification of the resulting overall model.

This note is structured as follows. In Section II, the notation used is presented. In Section III, contractions are defined and their properties are studied. In Section IV, an efficient algorithm for computing a set of good reduced-order approximations by contraction is presented. In

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Section V, a quantitative measure to choose the best among a set of reduced order approximations is given.

II. Background

An FSM [3], [4] is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where: Q is a finite-state set, Σ is a finite alphabet of symbols, $\delta : Q \times \Sigma \rightarrow 2^Q$ is the transition relation, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is a set of final states. The transition relation δ is usually extended to apply to a state and a string, rather than a state and a symbol.

Let $w = a_1 a_2 \cdots a_r \in \Sigma^*$ and $q' \in \delta(q, w)$. Then, the following is a legal move of M: $m(q, w) = q[a_1\rangle q_1[a_2\rangle \cdots q_{r-1}[a_r\rangle q' = q[w\rangle q'$ and we define $m_Q(q, w) = \{q_1, \cdots, q_{r-1}\}$.

An FSM is said to be *deterministic* (DFSM) if the transition relation is such that $\delta(q, a)$ is a singleton set or is not defined.

The language generated by a FSM M is the set of all strings w generated with a move that starts from the initial state and reaches a final state, i.e.,

$$L(M) = \{w \in \Sigma^* | \delta(q_0, w) \cap F \neq \emptyset\}.$$

Note that the above definitions are slightly different from classic definitions of automata, but are consistent with the modern discrete event systems terminology. As an example, in the classic definition of deterministic automata it is required that $\delta(q, a)$ be defined for all $q \in Q$ and for all $a \in \Sigma$.

Note also that in the discrete event system approach [7], there are usually two different notions of languages. The *marked behavior* is identical to the language $L(M)$ defined above. The *closed behavior* is defined as the set of strings generated with a move that starts from the initial state and reaches any state of M. Without any loss of generality, this note will only consider marked languages, since any closed language can be considered as a marked language if one lets the set of final states F be identical to the set of all states Q.

A DFSM $M = (Q, \Sigma, \delta, q_0, F)$ with n states is said to be *minimal* [5], [6] if there does not exist a DFSM $M' = (Q', \Sigma, \delta', q'_0, F')$ with $n' < n$ states such that $L(M') = L(M)$. Note that in the classic definition of minimal FSM there is always a "dump" state that can be reached by all strings that cannot be continued into a string in $L(M)$. Since we do not require that $\delta(q, a)$ be defined for all $q \in Q$ and for all $a \in \Sigma$, a minimal DFSM according to our definition will be *reachable* (i.e., there is a path from q_0 to any other state) and *coreachable* (i.e., there is a path from any state q to a state in F).

Let M be a minimal DFSM with n states. It is not possible to find a DFSM M' with $n' < n$ states that generates $L(M)$. However, we can look for an M' with $n' < n$ states that generates $L(M') \supset L(M)$ as a way to approximate M.

Definition 1: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a minimal DFSM with n states.

1) A language $L_a \subseteq \Sigma^*$ is an *approximation* of $L(M)$ if $L_a \supset L(M)$.
2) An approximation L_a is *order n_a implementable* if there exists a minimal DFSM M_a with $n_a < n$ states such that $L_a = L(M_a)$. We also say that M_a implements L_a and that it is an *approximation of order n_a* of M.
3) An order n_a implementable approximation L_a of $L(M)$ is *infimal* if there does not exist another DFSM M' with $n' \leq n_a$ states such that $L_a \supset L(M') \supset L(M)$. If M_a implements L_a, we say that M_a is an *infimal approximation of order n_a* of M.

Infimal approximations of a minimal DFSM M are the best approximations, in the sense that, compatibly with the state space size limitation, their behavior contains the behavior of M and is as close as possible to it.

III. Contractions

Given a minimal DFSM M with n states how can one find an infimal approximation of order $n' < n$? One possibility is that of computing all DFSMs with n' states over the same alphabet Σ of M and of looking for those that satisfy the definition of infimal approximations. However, this approach is clearly infeasible in light of the following proposition.

Proposition 1: There are $(n'+1)^{m \cdot n'} \cdot 2^{n'}$ different DFSMs with n' states and alphabet Σ of cardinality m.

Proof: According to the definition of DFSM given in the previous section, for all $q \in Q$ and all $a \in \Sigma$ there are $n' + 1$ possible choices of $\delta(q, a)$, keeping in mind that it may be undefined. Thus, there are $(n'+1)^{m \cdot n'}$ different possible choices of δ. Finally, since F is a subset of Q, there are $2^{n'}$ different possible choices of F. □

We will explore the possibility of using contractions, whose structure can be easily computed, as means of finding approximations of a given minimal DFSM M.

Definition 2: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFSM and let $q_i, q_j \in Q$, with $q_i \neq q_j$. The (i, j)*-contraction* of M is the FSM $M_{i,j}$ obtained from M by merging states q_i and q_j. Formally, $M_{i,j} = (Q', \Sigma, \delta', q'_0, F')$, where

1) the state set is $Q' = Q \cup \{q_{new}\} \backslash \{q_i, q_j\}$;
2) the transition relation is shown in the equation at the bottom of the page;
3) the initial state is

$$q'_0 = \begin{cases} q_0, & \text{if } q_0 \in Q \cap Q' \\ q_{new}, & \text{otherwise;} \end{cases}$$

4) the set of final states is

$$F' = \begin{cases} F, & \text{if } F \subseteq Q' \\ F \cup \{q_{new}\} \backslash \{q_i, q_j\}, & \text{otherwise.} \end{cases}$$

Note that $M_{i,j}$ may well be nondeterministic even if M is a DFSM. In Fig. 1, a DFSM M and its three possible contractions are shown. $M_{0,1}$ is nondeterministic and nonminimal; $M_{0,2}$ and $M_{1,2}$ are deterministic and minimal.

Let us consider some properties of contractions.

Lemma 1: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFSM and let $M_{i,j} = (Q', \Sigma, \delta', q'_0, F')$ be its (i, j)-contraction. Then

$$L(M_{i,j}) = L(M) \cup [(L_0^i \cup L_0^j) L_{i,j}^* (L_i \cup L_j)]$$

$$\delta'(q, a) = \begin{cases} \delta(q, a), & \text{if } q \in Q \cap Q' \wedge \delta(q, a) \in Q \cap Q' \\ q_{new}, & \text{if } q \in Q \cap Q' \wedge \delta(q, a) \in \{q_i, q_j\} \\ \delta(q_i, a) \cup \delta(q_j, a), & \text{if } q = q_{new} \wedge \delta(q_i, a) \cup \delta(q_j, a) \subseteq Q \cap Q' \\ \delta(q_i, a) \cup \delta(q_j, a) \cup \{q_{new}\} \backslash \{q_i, q_j\}, & \text{otherwise.} \end{cases}$$

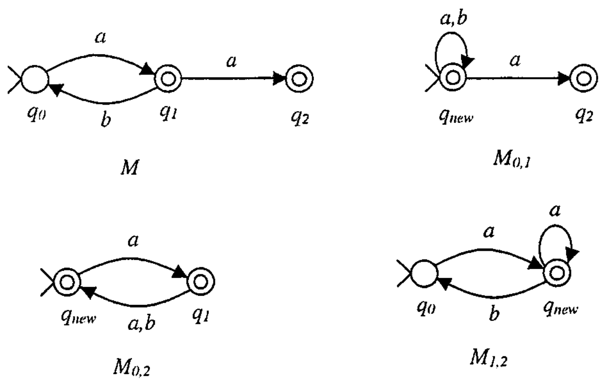

Fig. 1. An FSM M and its contractions.

where

$$\begin{cases} L_h^k = \{w \in \Sigma^* | \delta(q_h, w) = q_k; q_i, q_j \notin m_Q(q_h, w)\} \\ L_h = \{w \in \Sigma^* | \delta(q_h, w) \in F; q_i, q_j \notin m_Q(q_h, w)\} \\ L_{i,j} = (L_i^i \cup L_i^j \cup L_j^i \cup L_j^j). \end{cases}$$

Proof: We will just give a sketch of the proof. First, note that from the definition of contraction, it follows that, for all w such that $q_{new} \notin m_{Q'}(q_0', w)$:

$$\delta'(q_0', w) = q_{new} \iff w \in (L_0^i \cup L_0^j)$$

and for all w such that $q_{new} \notin m_{Q'}(q_{new}, w)$

$$\delta'(q_{new}, w) = q_{new} \iff w \in L_{i,j}$$
$$\delta'(q_{new}, w) \in F' \iff w \in (L_i \cup L_j).$$

Since a word $w \in L(M_{i,j})$ is generated either with a move $m(q_0', w) = q_0'[w\rangle q_f' \in F'$, where $q_{new} \notin m_{Q'}(q_0', w)$, or with a move $m(q_0', w) = q_0'[w_0\rangle q_{new} \cdots [w_{r-1}\rangle q_{new}[w_r\rangle q_f' \in F'$, where $q_{new} \notin m_{Q'}(q_0', w_0)$ and for all $k > 0$, $q_{new} \notin m_{Q'}(q_{new}, w_k)$, it is possible to prove the result of the lemma. □

Proposition 2: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFSM and let $M_{i,j} = (Q', \Sigma, \delta', q_0', F')$ be its (i, j)-contraction. Then, $L(M_{i,j}) \supseteq L(M)$. Also, if M is a minimal DFSM, then $L(M_{i,j}) \supset L(M)$.

Proof: The fact that $L(M_{i,j}) \supseteq L(M)$ trivially follows from Lemma 1.

If M is minimal, then states q_i and q_j are distinguishable, i.e., there must exist a string w_i such that, say, $\delta(q_i, w_i)$ is in F while $\delta(q_j, w_i)$ is not defined or is not in F. Now, let $w_{0,j}$ be a string such that $\delta(q_0, w_{0,j}) = q_j$. Then, $w_{0,j}w_i \notin L(M)$ while by Lemma 1 $w_{0,j}w_i \in L_{0,j}L_i \subseteq L(M_{i,j})$. □

According to the above proposition, the languages generated by contractions of a DFSM M are approximations of $L(M)$.

Example 1: The requirement that M be minimal in Proposition 2 can be explained by the following example. Fig. 2 shows a DFSM M that is not minimal and its contraction $M_{0,1}$. It can be seen that $L(M) = L(M_{0,1}) = a^*$.

Example 2: Not all languages generated by contractions are infimal approximations. Consider the minimal DFSM M in Fig. 1 and its three contractions. The language generated by $M_{0,1}$ is $L(M_{0,1}) = \Sigma^*$, i.e., it is a superset of the languages generated by the contractions $M_{0,2}$ and $M_{1,2}$. In this case, however, it is possible to prove that $M_{0,2}$ and $M_{1,2}$ are the only infimal approximations of M of order 2. To prove this one may construct all approximations of M of order 2.

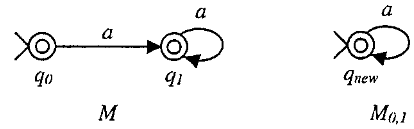

Fig. 2. A nonminimal FSM M and its contraction.

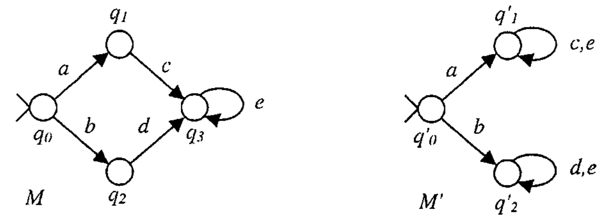

Fig. 3. A minimal FSM M with four states and an approximation of order 3.

Example 3: Not all infimal approximations of order $n-1$ of a minimal DFSM M with n states are contractions. Consider the DFSM M with four states and the DFSM M' with three states in Fig. 3. M' is an approximation of M since $L(M') \supset L(M)$ but it can be easily checked that it is not a contraction, because its language is not a superset of any contraction of M. Hence, there exists an infimal approximation of M of order 3 that is not a contraction. Note, however, that in this case it can also be shown that for all q_i, q_j, $L(M_{i,j})$ is not a superset of $L(M')$. Hence, one cannot conclude that the contractions of M are not infimal approximations.

Contractions are good candidates for infimal approximations of a minimal DFSM M. There are some cases in which it is possible to prove that any implementable approximation of $L(M)$ is a superset of a language generated by some contraction of $L(M)$.

Theorem 1: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a minimal DFSM with n states and let $M' = (Q', \Sigma, \delta', q_0', F')$ be a minimal DFSM with $n' < n$ such that $L(M') \supset L(M)$. Let $h : Q \to 2^{Q'}$ be the mapping defined by

$$\begin{cases} q_0' \in h(q_0) \\ q' \in h(q), \quad \text{if } \tilde{q}' \in h(\tilde{q}) \wedge \delta(\tilde{q}, a) = q \\ \qquad\qquad\quad \wedge\, \delta'(\tilde{q}', a) = q'. \end{cases}$$

If $h(q)$ is a singleton set for all $q \in Q$, then there exists an (i, j)-contraction of M such that

$$L(M') \supseteq L(M_{i,j}) \supset L(M).$$

Proof: Since $h(q)$ is a singleton set and $n > n'$, there must exist two states $q_i, q_j \in Q$ such that $h(q_i) = h(q_j) = q'$. Then, it is possible to prove that $L(M') \supseteq L(M_{i,j})$.

In fact, by the definition of h and the fact that $L(M') \supset L(M)$ it follows that if $\delta(q, w) = \tilde{q}$ then $\delta'(h(q), w) = h(\tilde{q})$ while $h(F) \subset F'$.

Hence, with the notation of Lemma 1

$$\forall w \in L_0^i \cup L_0^j, \delta'(q_0', w) = q'$$
$$\forall w \in L_i^i \cup L_i^j \cup L_j^i \cup L_j^j, \delta'(q', w) = q'$$
$$\forall w \in L_i \cup L_j, \delta'(q', w) \in F'$$

and any string in the set $L(M_{i,j})$, whose expression is given in Lemma 1, can also be generated by M'. □

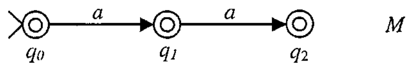

Fig. 4. A minimal DFSM M in Example 4.

Note 1: There are DFSMs M such that, regardless of the structure of M', the image of $h(q)$, as defined in the above theorem, is a singleton set. As an example, let M be a DFSM with a tree-like graph. Since there is only one path from the initial state to any other state and since M' is deterministic, $h(q)$ can only assume a single value. Thus, for this class of DFSMs it follows from Theorem 1 that if all languages generated by contractions are implementable then all infimal approximations of order $n-1$ of $L(M)$ are contractions.

The author's feeling is that the implementable languages generated by contractions of a minimal DFSM M are almost always infimal approximations of $L(M)$ because no counterexample has been found to disprove the following conjecture.

Conjecture 1: Let M be a minimal DFSM. Let

$$\mathcal{L} = \{L(M_{i,j}) | \nexists M_{h,k} \ni L(M_{i,j}) \supset L(M_{h,k})\}.$$

Then, all implementable languages in $\mathcal{L}$ are infimal approximations of $L(M)$.

IV. Implementing an Approximation

In the previous section, we have seen how to construct approximations of the language generated by a given minimal DFSM M by looking at its contractions.

We have also noted that a contraction is not always deterministic. Thus, to implement a contraction language we may have to convert a contraction $M_{i,j}$ into a deterministic FSM. The following examples will show several possible cases.

Example 4: In this example we consider contractions that are nonminimal. Consider the minimal DFSM with three states in Fig. 4. It is easy to see that all its contractions generate the language $L = a^*$, that can be generated by a single state DFSM. Since all contractions of M have two states they are not minimal. Note that $M_{0,1}$ is nondeterministic, while $M_{0,2}$ and $M_{1,2}$ are deterministic.

Example 5: In this example, we show that not all languages generated by contraction are implementable. Consider the minimal DFSM M with five states in Fig. 5. The contraction $M_{0,2}$ is not deterministic. When we compute the minimal DFSM that generates $L(M_{0,2})$, we obtain the DFSM $M^D_{0,2}$ that has six states.

The following algorithm can be used to compute a set $\mathcal{M}$ of good approximations of a minimal DFSM.

Algorithm 1: Let M be a minimal DFSM with n states.

1) Construct the set $\mathcal{M}^c$ of all contractions of M.
2) Let $\mathcal{M}^m$ be the set constructed as follows. For all contractions $M_{i,j} \in \mathcal{M}^c$:
   a) If $M_{i,j}$ is deterministic let $M^D_{i,j} = M_{i,j}$, else let $M^D_{i,j}$ be a DFSM equivalent to $M_{i,j}$.
   b) If $M^D_{i,j}$ is minimal let $M^m_{i,j} = M^D_{i,j}$, else let $M^m_{i,j}$ be a minimal DFSM equivalent to $M^D_{i,j}$.
   c) If the number of states of $M^m_{i,j}$ is $n_m < n$, let $M^m_{i,j} \in \mathcal{M}^m$.
3) Let $\mathcal{M} = \{M' \in \mathcal{M}^m | \nexists M'' \in \mathcal{M}^m \ni L(M') \supset L(M'')\}$. $\mathcal{M}$ is a set of approximations of M of order less than n.

Following are some comments on the complexity of the algorithm. In Step 1), there are $\binom{n}{2} = (n(n-1))/2$ contractions.

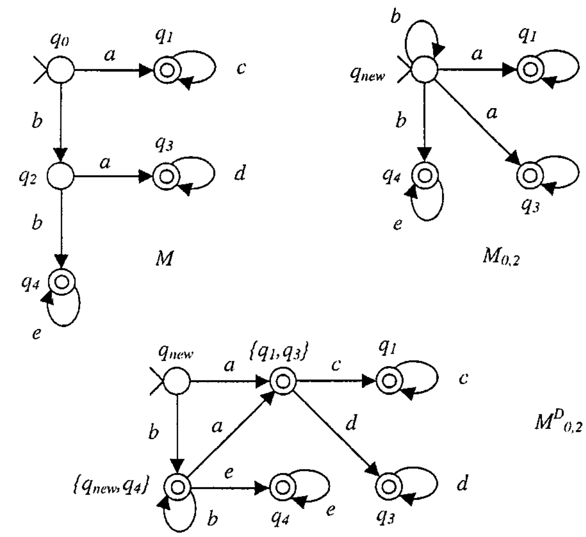

Fig. 5. A DFSM M and its contraction $M_{0,2}$ whose language cannot be implemented.

Step 2a) is the computationally hardest step. In fact, a DFSM M^D equivalent to a nondeterministic one M with n states may have up to 2^n states [3]. This means that in general the "determinization" cannot be done in polynomial time or space.

In Step 2b), the "minimization" of a DFSM with n' states can be done with an $n' \log n'$ algorithm given in [2].

In Step 3), one can use the algorithm given in [1, p. 144] to check if $L(M_1) \subseteq L(M_2)$. If M_1 has n_1 states and M_2 has n_2 states the complexity of the algorithm is $nG(n)$, where $n = n_1 + n_2$ and $G(n) \leq 5$ for $n \leq 2^{65\,536}$.

V. Choosing the Best Approximation

In this section, we consider the following problem. Given a set $\mathcal{M}$ of approximations of a given minimal DFSM M and a finite set of observed strings $L_o \subseteq L(M)$, choose among all FSMs in $\mathcal{M}$ the best approximation relative to the observed behavior, i.e., the approximation M' that maximizes a suitable function $f(L_o, M')$.

First of all, given $M' = (Q', \Sigma, \delta', q'_0, F')$ we define two functions $\nu, \mu : Q' \to \mathbb{N}$. The first one is such that $\nu(q') = 1$ if $q' \in F'$ (i.e., if it is a final state), else $\nu(q') = 0$. The second one is such that $\mu(q') = |\{a \in \Sigma^* | \delta'(q', a) \text{ is defined }\}|$, i.e., it counts the number of events enabled at q'.

If we have no additional knowledge, we may assume that at each step while generating a string w and being in a state q', M' may choose with equal probability to accept the string generated so far (if q' is a final state) or to continue, executing one of the events enabled at q'. The total number of choices at each state is thus $\nu(q') + \mu(q')$.

Thus, let $w = a_1 a_2 \cdots a_r$ be generated by M' with the move

$$m(q'_0, w) = q'_0[a_1\rangle q'_1[a_2\rangle \cdots q'_{r-1}[a_r\rangle q'_r.$$

We define a *merit function*

$$f(w, M') = \prod_{i=0}^{r} \frac{1}{\nu(q'_i) + \mu(q'_i)}$$

whose value is a measure of the likelihood that w is generated by M'.

Example 6: Consider the DFSM M in Fig. 1 and its two approximations $M_{0,2}$, and $M_{1,2}$. The string $w_1 = (ab)^k a$ is more likely to be generated by $M_{1,2}$ since

$$f(w_1, M_{0,2}) = \left(\frac{1}{2} \cdot \frac{1}{3}\right)^k \cdot \frac{1}{2} = \frac{1}{2 \cdot 6^k}$$

while

$$f(w_1, M_{1,2}) = \left(1 \cdot \frac{1}{3}\right)^k \cdot 1 = \frac{1}{3^k}.$$

On the contrary, the string $w_2 = aa^{2k}$ for $k > 1$ is more likely to be generated by $M_{0,2}$ since

$$f(w_2, M_{0,2}) = \frac{1}{2} \cdot \left(\frac{1}{3} \cdot \frac{1}{2}\right)^k = \frac{1}{2 \cdot 6^k}$$

while

$$f(w_2, M_{1,2}) = 1 \cdot \left(\frac{1}{3} \cdot \frac{1}{3}\right)^k = \frac{1}{9^k}.$$

The next proposition shows that f is a good measure for choosing among approximations in the sense that it tends to give higher rating to infimal approximations.

Proposition 3: Let $M = (Q, \Sigma, \delta, q_0, F)$ and $M' = (Q', \Sigma, \delta', q_0', F')$ be DFSMs such that $L(M) \subset L(M')$. Then, for all $w \in L(M)$, $f(w, M) \geq f(w, M')$.

Proof: Let $w = a_1 a_2 \cdots a_r$ be generated by M with the move $q_0[a_1\rangle q_1 \cdots [a_r\rangle q_r$, and by M' with the move $q_0'[a_1\rangle q_1' \cdots [a_r\rangle q_r'$. Since $L(M) \subset L(M')$, it follows that:

1) $q_i' \in F'$ if $q_i \in F$, i.e., $\nu(q_i') \geq \nu(q_i)$;
2) $\delta(q_i', a)$ is defined if $\delta(q_i, a)$ is defined, i.e., $\mu(q_i') \geq \mu(q_i)$.

Hence, $f(w, M) \geq f(w, M')$. □

The merit function f can be extended to set of strings. If $L \subseteq L(M')$, we define

$$f(L, M') = \prod_{w \in L} f(w, M').$$

Thus, given a set $\mathcal{M}$ of approximations of a given minimal DFSM M and a finite set of observed strings L_o, we say that the best approximation of M with respect to f and L_o is the DFSM $M' \in \mathcal{M}$ such that

$$f(L_o, M') = \max_{M'' \in \mathcal{M}} [f(L_o, M'')].$$

Different merit functions could be used if we assume that some knowledge on the probability of occurrence of different events in Σ^* is known.

VI. Conclusion

This note has presented introductory work on the model reduction of discrete event systems represented by FSMs.

It was shown how a set of good—but possibly not infimal—approximations of a given minimal DFSM M can be computed efficiently by looking at contractions of M. In some particular case, it is also possible to prove that the approximations thus constructed are infimal.

This note has also discussed how, given a set of approximations of an FSM M, it is possible to define a merit function to choose the best approximation with respect to a given observed behavior $L_o \subseteq L(M)$.

The approach presented in this note leaves open some interesting problems. First, we do not know if the conjecture presented in Section III is true; it should be possible to prove it or to find a counterexample to disprove it. Second, it may be interesting to try to apply the contraction technique to other graphical models of discrete event systems such as Petri nets.

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