

# On the Choice of Suboptimal Monitors for Supervisory Control of Petri Nets\*

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## Abstract

*Recent results in the literature have provided an efficient control synthesis technique for the problem of enforcing generalized mutual exclusion constraints on place/transition nets. With this technique both the plant and the controller are described by Petri nets in order to have an useful linear algebraic model for control analysis and synthesis. The synthesis is not computation demanding since it involves only matrix multiplications. Moreover the method has been shown to be maximally permissive in the case of controllable specifications, otherwise the controller may be suboptimal and its structure may not be unique. This paper investigates on and provides an algorithm to compute these control structures and two criteria of suboptimality.*

## 1 INTRODUCTION

Let us consider a PN system  $\langle N, \mu_0 \rangle$  with  $m$  places, whose set of reachable markings is  $R(N, \mu_0) \subseteq \mathbb{N}^m$ . Assume we are given a set of legal markings  $\mathcal{L} \subseteq \mathbb{N}^m$ , and consider the basic control problem of designing a supervisor that restricts the reachability set of plant in closed loop to  $\mathcal{L} \cap R(N, \mu_0)$ . This is possible if and only if  $\mathcal{L}$  is controllable (and reachable). If  $\mathcal{L}$  is not controllable, we can consider the class of controllable subsets of  $\mathcal{L}$ , i.e., the class  $\Omega(\mathcal{L}) = \{\mathcal{K} \subseteq \mathcal{L} \mid \mathcal{K} \text{ is controllable}\}$ . The class  $\Omega(\mathcal{L})$  is not empty and closed under union, if the non-concurrency hypothesis holds, hence it admits a unique supremal element with respect to set inclusion. The element  $\mathcal{L}^\dagger = \sup \Omega(\mathcal{L})$ , called *supremal controllable subset*, is the “optimal” solution to this control problem.

Of particular interest are those PN state-based control problem where the set of legal markings  $\mathcal{L}$  is expressed by a set of  $n_c$  linear inequality constraints called Generalized Mutual Exclusion Constraint (GMEC). In this case we write  $\mathcal{L} = \mathcal{M}(\mathbf{L}, \mathbf{k}) \equiv \{\mu \in \mathbb{N}^m \mid \mathbf{L}\mu \leq \mathbf{k}\}$  to denote that  $\mathcal{L}$  is expressed by the GMEC  $(\mathbf{L}, \mathbf{k})$  with  $\mathbf{L} \in \mathbb{Z}^{n_c \times m}, \mathbf{k} \in \mathbb{Z}^{n_c}$ . Problems of this kind have been considered by several authors [2, 7, 5]. This special structure of the legal set has the advantage that if  $\mathcal{L}$  is controllable then the supervisor for this class of problems takes the

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form of as many places, called *monitors*, as there are constraints. Thus if the matrix  $\mathbf{L}$  has  $n_c$  rows, the supervisor will consist of  $n_c$  monitor places, each of which has arcs going to and coming from some transitions of the plant net. This control structure can be easily analyzed and implemented.

Let us assume, however, that  $\mathcal{L}$  is uncontrollable. The counterpart on the controller structure is that one of the monitors associated to this GMEC has arcs going to uncontrollable transitions, i.e., it may be blocking an uncontrollable transition. Following the general approach outlined above, we have to compute the set  $\mathcal{L}^\uparrow$ , but unfortunately, as shown by Giua et al. [2], it may well be the case that this set cannot be expressed by a set of linear inequalities, i.e., the corresponding supervisor does not have a monitor-based structure. Li and Wonham [5] showed that if the plant net belongs to the special class of TS2 nets then  $\mathcal{L}^\uparrow$  is guaranteed to be expressed by a set of  $n_c$  linear inequalities. Giua et al. [2] showed that if the plant net is safe then  $\mathcal{L}^\uparrow$  is guaranteed to be expressed by a set of  $n'_c$  linear inequalities, where  $n'_c$ , however, may be very large (it may be of the same order of the cardinality of the reachability set).

This problem motivated Moody et al. [6, 7] to consider as acceptable a further restriction of the reachability set. Given an uncontrollable legal marking set  $\mathcal{L}$  expressed by  $n_c$  constraints, one may define the set  $\Omega_{n_c}(\mathcal{L}) = \{\mathcal{K} \subseteq \mathcal{L} \mid \mathcal{K} \text{ is controllable, } \exists \mathbf{L}' \in \mathbb{Z}^{n_c \times m}, \mathbf{k}' \in \mathbb{Z}^{n_c} : \mathcal{K} = \mathcal{M}(\mathbf{L}', \mathbf{k}')\}$  of *controllable* and *expressed by a set of  $n_c$  linear inequalities* subsets of  $\mathcal{L}$ . In [6] a procedure was also given that leads to compute an element  $\mathcal{K} \in \Omega_{n_c}(\mathcal{L})$ , i.e., to compute a constraint  $(\mathbf{L}', \mathbf{k}')$  with  $\mathbf{L}' \in \mathbb{Z}^{n_c \times m}$ , and its corresponding monitor structure, such that  $\mathcal{K} = \mathcal{M}(\mathbf{L}', \mathbf{k}')$ . We note that in this approach one restricts the reachability set of the plant in closed loop to be within  $\mathcal{K} \subset \mathcal{L}^\uparrow$ , i.e., one may prevent the closed loop system from reaching some perfectly legal marking. One gains, however, in simplicity because the controller takes a simple structure of  $n_c$  monitors.

In [1] it has been shown formally show that the class  $\Omega_{n_c}(\mathcal{L})$  is not empty and not closed under union. Hence a supremal element exists but it is not necessarily unique.

In this paper we further pursue the investigation along these lines and present the following results.

- We give an algorithm to construct a parameterization of all monitors corresponding to *supremal* elements of  $\Omega_{n_c}(\mathcal{L})$ . This parameterization takes the form of a unique control net incidence matrix that depends on the value of the parameters subject to a linear equations system.
- We discuss a performance measures to choose among these supremal elements. In fact, since the elements of  $\Omega_{n_c}(\mathcal{L})$  cannot be ordered by subset inclusions, the  $\subseteq$  criterion of optimality is meaningless. We consider instead the cardinality of the reachability set allowed by each monitor. An heuristic criterion, based on transition firings, is also given. We will show that in many cases these criteria give similar results.

## 2 BACKGROUND

A place/transition (P/T) net is a structure  $N = (P, T, I, O)$  where:  $P$  is a set of  $m$  *places* represented by circles;  $T$  is a set of  $n$  *transitions* represented by bar;  $P \cap T = \emptyset$ ,  $P \cup T \neq \emptyset$ ;  $I : P \times T \rightarrow \mathbb{N}$  is the input function that specifies the arcs directed from places to transitions,

with  $\mathbb{N}$  is the set of non-negative integers;  $O : P \times T \rightarrow \mathbb{N}$  is the output function that specifies the arcs directed from transitions to places. The *preset* and *postset* of a place are respectively:  $\bullet p = \{t \in T \mid O(p, t) > 0\}$  and  $p^\bullet = \{t \in T \mid I(p, t) > 0\}$ .

A *marking* is a  $m \times 1$  vector  $\mu : P \rightarrow \mathbb{N}$  that assigns to each place of a P/T net a non-negative integer number of tokens, represented by black dots. A transition  $t \in T$  is enabled at a marking  $\mu$  iff  $\mu \geq I(\cdot, t)$ . If  $t$  is enabled, then  $t$  may fire yielding a new marking  $\mu'$ . The notation  $\mu[t > \mu'$  will denote that an enabled transition  $t$  may fire at  $\mu$  yielding  $\mu'$ .  $\mathbb{N}^m$  will denote the set of all possible markings that may be defined on the net. A *firing sequence* from  $\mu_0$  is a (possibly empty) sequence of transitions  $\sigma = t_1 \dots t_k$  such that  $\mu_0[t_1 > \mu_1[t_2 > \mu_2 \dots [t_k > \mu_k$ . A P/T system or net system  $\langle N, \mu_0 \rangle$  is a P/T net  $N$  with an initial marking  $\mu_0$ . A marking  $\mu$  is reachable in  $\langle N, \mu_0 \rangle$  iff there exists a firing sequence  $\sigma$  such that  $\mu_0[\sigma > \mu$ . Given a net system  $\langle N, \mu_0 \rangle$  the set of reachable markings (also called *reachability set* of the net) is denoted  $R(N, \mu_0)$ . If a marking  $\mu$  is reachable in  $\langle N, \mu_0 \rangle$  by firing a sequence  $\sigma$ , then the following state equation is satisfied

$$\mu = \mu_0 + D\sigma \quad (1)$$

where  $D = O - I$  is the  $m \times n$  incidence matrix of the net  $N$  and  $\sigma : T \rightarrow \mathbb{N}$  is a  $n \times 1$  vector called *firing-vector* of the net.  $\sigma(t)$  represents the number of times that a transition  $t$  appears in  $\sigma$ . The set of markings  $\mu$  such that there exists an integer vector  $\sigma \geq 0$  satisfying the previous state equation is called *potentially reachable set* and is denoted  $PR(N, \mu_0)$ . Note that  $PR(N, \mu_0) \supseteq R(N, \mu_0)$ . For special classes of nets such as acyclic nets  $PR(N, \mu_0) = R(N, \mu_0)$ .

A single generalized mutual exclusion constraint (GMEC) is a couple  $(l, k)$  where  $l : P \rightarrow \mathbb{Z}$  is a  $1 \times m$  weight vector and  $k \in \mathbb{N}$ . The support of  $l$  is the set  $Q_l = \{p \in P \mid l(p) \neq 0\}$ . Given the net system  $\langle N, \mu_0 \rangle$ , a GMEC defines a set of markings that will be called *legal markings*:  $\mathcal{M}(l, k) = \{\mu \in \mathbb{N}^m \mid l\mu \leq k\}$ . The markings that are not legal are called *forbidden markings*. A set of GMEC  $(L, k)$ , with  $L = [l_1, l_2, \dots, l_{n_c}]^T$  and  $k = [k_1, k_2, \dots, k_{n_c}]^T$ , will define the *legal markings set*  $\mathcal{M}(L, k) = \{\mu \in \mathbb{N}^m \mid L\mu \leq k\}$ . A controlling agent, called supervisor, must ensure the forbidden markings will be not reached. So the set of legal markings under control is  $\mathcal{M}_c(L, k) = \mathcal{M}(L, k) \cap R(N, \mu_0)$ .

The set of the transitions  $T$  of a net  $N$  is now assumed to be partitioned into two disjoint subsets:  $T_u$  the set of the *uncontrollable transitions* and  $T_c$  the set of *controllable transitions*. The occurrence of a controllable transition may be disabled, while the occurrence of an uncontrollable transition cannot be disabled. In this case it is useful to consider the net  $N_u$  obtained from the net  $N$  eliminating the uncontrollable transitions, whose incidence matrix is denoted by  $D_{uc}$ .

### GMEC forcing of controllable specifications

We say that a set  $\mathcal{L} \subseteq \mathbb{N}^m$  of legal markings is controllable with respect to a PN system  $\langle N, \mu_0 \rangle$  with uncontrollable subnet  $N_u$  if  $\bigcup_{\mu \in \mathcal{L} \cap R(N, \mu_0)} R(N_u, \mu) \subseteq \mathcal{L}$ . If  $\mathcal{L}$  is controllable, it has been shown [7] that the Petri net controller that enforces  $(L, k)$  has the incidence matrix  $D_c \in \mathbb{Z}^{n_c \times n}$  given by

$$D_c = -LD_p \quad (2)$$

and the initial marking of the controller  $\mu_{c0} \in \mathbb{N}^{n_c}$  is given by

$$\mu_{c0} = \mathbf{k} - \mathbf{L}\mu_{p0} \quad (3)$$

where  $\mu_{p0} \in \mathbb{N}^m$  is the initial marking of the plant. The controller exists iff the initial marking is a legal marking, i.e.

$$\mathbf{k} - \mathbf{L}^-_{p0} \geq \mathbf{0} \quad (4)$$

The controller so constructed is maximally permissive, i.e. it prevents only transitions firings that yield forbidden markings. The control net has  $n_c$  places (one for each constraint). Each place of the control net it is called *monitor place*.

### GMEC forcing of uncontrollable specifications

Because of the occurrence of an uncontrollable transition  $t_u$  enabled at a certain legal marking  $\bar{\cdot}$ , a forbidden marking  $\bar{\cdot}'$  may be reached, so it is necessary avoid also the set of markings  $\mathcal{M}_{fu}(\mathbf{L}, \mathbf{k}) = \{\bar{\cdot} \in \mathbb{N}^m \mid \bar{\cdot}[\mathbf{c}e > \bar{\cdot}', \bar{\cdot}' \notin \mathcal{M}(\mathbf{L}, \mathbf{k}), \mathbf{c}e \in T_u^*]\}$ . So in presence of uncontrollable transition the set of legal markings under control will be  $\mathcal{M}_c(\mathbf{L}, \mathbf{k}) = (\mathcal{M}(\mathbf{L}, \mathbf{k}) \cap R(N, \bar{\cdot}_0)) \setminus \mathcal{M}_{fu}(\mathbf{L}, \mathbf{k})$ . Given a set  $X$ ,  $|X|$  will denote its cardinality. As a result of the presence of uncontrollable transition it is  $|\mathcal{M}_c(\mathbf{L}, \mathbf{k})| \leq |\mathcal{M}(\mathbf{L}, \mathbf{k})|$ , i.e. the cardinality of the set of legal markings is decreased.

The potentially reachable marking set of the plant from a marking  $\bar{\cdot}$  by firing only uncontrollable transition is

$$PR_u(N, \bar{\cdot}) = \{\bar{\cdot}' \in \mathbb{N}^m \mid \bar{\cdot}' = \bar{\cdot} + \mathbf{D}_{uc}\mathbf{c}e_u, \mathbf{c}e_u \geq \mathbf{0}\}$$

where  $\mathbf{D}_{uc} \in \mathbb{Z}^{m \times n_{uc}}$  is the incidence matrix of the net  $N_{uc}$ . If there exists a marking  $\bar{\cdot}' \in PR_u(N, \bar{\cdot})$  that is not legal then  $\bar{\cdot}$  is a potentially forbidden marking. The set of potentially forbidden markings because of firing only uncontrollable transitions is

$$\mathcal{P}_{fu}(\mathbf{L}, \mathbf{k}) = \{\bar{\cdot}_p \in \mathbb{N}^m \mid \exists \mathbf{c}e_u \geq \mathbf{0}, \mathbf{L}^-_{\bar{\cdot}_p} + \mathbf{L}\mathbf{D}_{uc}\mathbf{c}e_u > \mathbf{k}\}$$

Note that  $\mathcal{P}_{fu} \supseteq \mathcal{M}_{fu}$ . If the net system is acyclic then  $\mathcal{P}_{fu} = \mathcal{M}_{fu}$ . It may be checked if a marking  $\mu_p$  belongs to  $\mathcal{P}_{fu}$  by the predicate

$$\mathbf{L}^-_{\bar{\cdot}_p} + \mathbf{L}\mathbf{D}_{uc}\mathbf{c}e_u^* \leq \mathbf{k}$$

where  $\mathbf{c}e_u^*$  is the solution of the ILP  $\forall \bar{\cdot}_p \geq \mathbf{0}$

$$\begin{aligned} & \max_{\sigma_u} \mathbf{L}\mathbf{D}_{uc}\mathbf{c}e_u \\ \text{s.t. } & \begin{cases} \mathbf{c}e_u \geq \mathbf{0} \\ \mathbf{D}_{uc}\mathbf{c}e_u \geq -\bar{\cdot}_p \end{cases} \end{aligned} \quad (5)$$

The solution  $\mathbf{c}e_u^*$  of the above ILP is not in general a linear function of  $\bar{\cdot}_p$ . So the set of markings that we enforce by monitor places is the subset  $\mathcal{M}'_c \subseteq \mathcal{M}_c$ . In addition it is possible to prove [2] that there may not exist a GMEC such that only the markings in the set  $\mathcal{M}_c$  are allowed. In this case a monitor-based solution will be suboptimal. In presence of uncontrollable transitions we have to satisfy the GMEC:

$$\mathbf{L}\mathbf{D}_{uc} \leq \mathbf{0} \quad (6)$$

Owing to (2), if (6) is not satisfied, the elements of the incidence matrix of the Petri net controller  $\mathbf{D}_c$  related to arcs between a monitor-place and an uncontrollable transition may be positive, while an uncontrollable transition cannot be disabled. It is possible [6] to transform  $(\mathbf{L}, \mathbf{k})$  in order to include (6).

**Proposition 1 (Moody, et al. [6])** *If we are able to find  $\mathbf{R}_1$  and  $\mathbf{R}_2$  satisfying*

$$\mathbf{R}_1 \in \mathbb{Z}^{n_c \times m}, \mathbf{R}_1^{-p} \geq \mathbf{0}, \forall^{-p} \quad (7)$$

$$\mathbf{R}_2 \in \mathbb{N}^{n_c \times n_c} \text{ positive definite, diagonal matrix} \quad (8)$$

and

$$\begin{aligned} [\mathbf{R}_1 \quad \mathbf{R}_2] \begin{bmatrix} \mathbf{D}_{uc} & \mathbf{1}^{-p_0} \\ \mathbf{L}\mathbf{D}_{uc} & \mathbf{L}^{-p_0} - \mathbf{k} - \mathbf{1} \end{bmatrix} \\ \leq [\mathbf{O}_{m \times n_{uc}} \quad -\mathbf{1}] \end{aligned} \quad (9)$$

with  $\mathbf{1}$   $n_c \times 1$  vector of 1's, then the controller computed as

$$\mathbf{D}_c = -\mathbf{L}'\mathbf{D}_p \quad (10)$$

$$\mathbf{1}^{-c_0} = \mathbf{k}' - \mathbf{L}'^{-p_0} \quad (11)$$

where

$$\mathbf{L}' = \mathbf{R}_1 + \mathbf{R}_2\mathbf{L} \quad (12)$$

$$\mathbf{k}' = \mathbf{R}_2(\mathbf{k} + \mathbf{1}) - \mathbf{1}. \quad (13)$$

will be able to ensure that the closed loop net system meet  $\mathbf{L}^{-p} \leq \mathbf{k}$ , (6) and

$$\mathbf{0} \leq (\mathbf{R}_1^{-p_0}) \leq \mathbf{R}_2(\mathbf{k} + \mathbf{1} - \mathbf{L}^{-p_0}) - \mathbf{1}. \quad (14)$$

i.e. that the initial marking is a legal marking.

### 3 AN ALGORITHM FOR THE CONSTRAINT TRANSFORMATION

It is possible to find  $\mathbf{R}_1$  and  $\mathbf{R}_2$  by matrix row operations working on the table of integers

$$\left[ \begin{array}{c|c|c} \mathbf{D}_{uc} & \mathbf{I} & \mathbf{O} \\ \hline \mathbf{N} & \mathbf{R}_1 & \mathbf{R}_2 \end{array} \right] \quad (15)$$

so that the positive elements in the  $\mathbf{L}\mathbf{D}_{uc}$  portion of the matrix  $\mathbf{M}$  become not positive. The aim of the constraint transformation, that will be shown, is to make zero the positive elements in  $\mathbf{L}\mathbf{D}_{uc}$ . If the net  $N_{uc}$  is cyclic the following algorithm may have a loop.

**Algorithm 1** *Constraint transformation matrixes computation.*

**begin**

$\mathbf{R}_1 := \mathbf{O}_{n_c \times m};$

$\mathbf{R}_2 := \mathbf{I}_{n_c \times n_c};$

**if not**  $(\mathbf{L}\mathbf{D}_{uc} \leq \mathbf{0})$  **then**

**begin**

$x := 0;$

$\mathbf{N} := \mathbf{L}\mathbf{D}_{uc};$

Consider the following  $(m + n_c) \times (n_{uc} + m + n_c)$   
table of integers:

$$\left[ \begin{array}{c|c|c} \mathbf{D}_{uc} & \mathbf{I} & \mathbf{O} \\ \hline \mathbf{N} & \mathbf{R}_1 & \mathbf{R}_2 \end{array} \right];$$

**repeat**

**begin**

Let  $B_s$  be the set of row indexes of negative  
elements in  $\mathbf{D}_{uc}(\cdot, s)$  and  $b_s$  its cardinality;

Choose  $\mathbf{N}(r, s) > 0$  such that  $b_s \leq b_t, \forall t \in \{0..n_c\}$ ;

**if**  $b_s \geq 1$  **then**

**begin**

$x := x + 1$ ;

(\*  $x$  is the algorithm step counter \*)

$\mathbf{C}(x, 1..m_{uc}) = \mathbf{0}_{1 \times m_{uc}}$ ;

$\mathbf{A}(1..m_{uc}, x) = \mathbf{0}_{m_{uc} \times 1}$ ;

Let  $\beta$  be the least common multiplier (l.c.m.)

between  $\mathbf{D}_{uc}(i, s)$  for  $i \in B_s$  and  $\mathbf{N}(r, s)$ ;

$\mathbf{N}(r, \cdot) := \beta \mathbf{N}(r, \cdot)$ ;

$\mathbf{R}_1 := \beta \mathbf{R}_1$ ;

$\mathbf{R}_2 := \beta \mathbf{R}_2$ ;

$\forall i \in B_s, \mathbf{C}(x, i) = \beta / \mathbf{D}_{uc}(i, r)$ ,

$\mathbf{A}(i, x) = \alpha_{x,i}$ ;

$\mathbf{b}(x) = -\mathbf{N}(r, s)$ ;

(\* The linear equations system  $\mathbf{CA} = \mathbf{b}$

that depends on  $\alpha_{x,i}$  ensures that

$\sum_{i \in B_s} \alpha_{l,i} \beta / \mathbf{D}_{uc}(i, r) = -\mathbf{N}(r, s)$ ,

$l = [1..x]$  are satisfied \*)

$\mathbf{N}(r, \cdot) := \mathbf{N}(r, \cdot) + \sum_{i \in B_s} \alpha_{x,i} \mathbf{N}(i, \cdot)$ ;

$\mathbf{R}_1(r, \cdot) := \mathbf{R}_1(r, \cdot) + \sum_{i \in B_s} \alpha_{x,i} \mathbf{I}(i, \cdot)$ ;

**end**

**else**

Constraint transformation is infeasible;

**end**

**until**  $\mathbf{N} \leq \mathbf{0}$

**end**

**end.**

The solutions of the above algorithm can be represented in the following compact form, giving  
a set of solutions to (9):

$$\mathbf{D}_c = -\mathbf{L}'\mathbf{D}_p, \mathbf{L}' = \mathbf{R}_1 + \mathbf{R}_2\mathbf{L}$$

$$\mathbf{R}_1 = [\sum_{i=1}^x \alpha_{i,1} \quad 0 \quad \dots \quad 0 \quad \sum_{i=1}^x \alpha_{i,l} \quad \dots \quad 0]$$

$$\mathbf{R}_2 \in \mathbb{N}^{n_c \times n_c} \text{ nonsingular, diag.}$$

$$\text{s.t. } \mathbf{CA} = \mathbf{b}$$

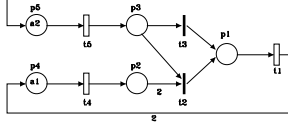


Figure 1: System in example 1.

where  $x$  is the number of steps of the algorithm, the nonzero elements of  $\mathbf{R}_1$  are those related to the  $m_{uc}$  places of the net  $N_{uc}$ ,  $\mathbf{C} \in \mathbb{N}^{x \times m_{uc}}$ ,  $\mathbf{A} \in \mathbb{N}^{m_{uc} \times x}$ ,  $\mathbf{b} \in \mathbb{N}^{n_c}$ .

A transformed constraint can be enforced iff

$$(\mathbf{R}_1 + \mathbf{R}_2 \mathbf{L})^{-p_0} \leq \mathbf{R}_2 (\mathbf{k} + \mathbf{1}) - \mathbf{1} \quad (16)$$

If an element of  $\mathbf{L} \mathbf{D}_{uc}$  is positive, the monitor based control net will have an output arc to the corresponding uncontrollable transition, because  $\mathbf{D}_c = -\mathbf{L} \mathbf{D}_{uc}$ . The aim of the constraint transformation is to move up to controllable transitions this arc, with the effect of including in the transformed constraint equation the places present in the path that connects the controllable transitions and the uncontrollable one.

The algorithm 1 is based on choosing as pivot the negative elements of the columns of  $\mathbf{D}_{uc}$  in order to make zero the positive elements of  $\mathbf{L} \mathbf{D}_{uc}$ . Each negative element in a column of  $\mathbf{D}_{uc}$  represents an input place of the uncontrollable transition related to this column. Because we have to choose a pivot for each column, it is possible to make as many choices as there are negative elements in each column. The effect of this choice is to add in the constraint equation the place related to this matrix element and to transform the arc of the control net directed to the uncontrollable transition into an arc connected to an input transition of this place.

Because in the algorithm shown the matrix row operations implemented are addition and multiplication by positive numbers the family of matrixes  $\mathbf{R}_1(\alpha_{1,1}, \dots, \alpha_{x,m_{uc}})$  and  $\mathbf{R}_2$  obtained meet (7) and (8), where  $m_{uc}$  is the number of places of the  $N_{uc}$  net. Once that it has been chosen the control structure, i.e. a certain solution  $\alpha_{1,1}^*, \dots, \alpha_{x,m_{uc}}^*$  of the  $x + n_c$  constraint equations system, the transformed constraint can be computed as in (12), (13). If the rows of the matrix  $[\mathbf{R}_1(\alpha_{1,1}^*, \dots, \alpha_{x,m_{uc}}^*) \quad \mathbf{R}_2]$  have a maximum common divider (m.c.d.) different from 1, it is useful to divide each row by its m.c.d. in order to simplify the implementation of the controller.

From each one of the constraint transformation a different control structure may be obtained by (10) and (11). Two of these control structures will be equivalent if these places have the same input transition.

**Example 1** The incidence matrix of the net in Fig. 1 is

$$\mathbf{D}_p = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 2 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The initial marking is  ${}^{-p_0} = [0 \ 0 \ 0 \ a_1 \ a_2]$ . If the control goal is  $\mu_1 \leq b$ , i.e.  $\mathbf{L} = [1 \ 0 \ 0 \ 0 \ 0]$ ,  $k = b$  we have that

$$\left[ \begin{array}{c|c|c} \mathbf{D}_{uc} & \mathbf{I} & \mathbf{O} \\ \hline \mathbf{N} & \mathbf{R}_1 & \mathbf{R}_2 \end{array} \right] =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By algorithm 1, if we choose to start making zero  $\mathbf{N}(1,2)$  we have that  $B_2 = \{3\}$  and it follows that  $\mathbf{R}_1 = [0 \ 0 \ 1 \ 0 \ 0]$ ,  $\mathbf{R}_2 = [1]$ . The unique controller structures is:

$$\begin{aligned} \mathbf{L}' &= [1 \ 0 \ 1 \ 0 \ 0], \quad k' = k \\ \mathbf{D}_c &= -\mathbf{L}'\mathbf{D}_p = [1 \ 0 \ 0 \ 0 \ -1] \\ \mu_{c0} &= k' - \mathbf{L}'^{-1}p_0 = b \end{aligned}$$

If we had started making zero the positive element  $\mathbf{N}(1,1)$  that has not the minimum number of negative elements in its column, we have obtained  $B_1 = \{2,3\}$  and the l.c.m. between  $\mathbf{N}(1,1)$  and  $\mathbf{D}_{uc}(i,1)$  for  $i \in B_1$  is 2 and so the first constraint equation is  $2\alpha_{1,2} + \alpha_{1,3} = 2$ . At this step of algorithm 1 we have:

$$= \left[ \begin{array}{c|c|c} \mathbf{D}_{uc} & \mathbf{I} & \mathbf{O} \\ \hline \mathbf{N} & \mathbf{R}_1 & \mathbf{R}_2 \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 - \alpha_{1,3} & 0 & \alpha_{1,2} & \alpha_{1,3} & 0 & 0 & 2 \end{bmatrix}$$

To make zero  $\mathbf{N}(1,2)$  we have that  $B_2 = \{3\}$  and it follows that the second constraint equation is  $\alpha_{2,3} = 1 - \alpha_{1,3}$ . The constraint transformation problem solutions are

$$\begin{aligned} \mathbf{R}_1 &= [0 \ \alpha_{1,2} \ \alpha_{1,3} + \alpha_{2,3} \ 0 \ 0] \\ \mathbf{R}_2 &= [2] \\ s.t. \quad &\begin{cases} 2\alpha_{1,2} + \alpha_{1,3} = 2 \\ \alpha_{2,3} = 2 - \alpha_{1,3} \end{cases} \end{aligned}$$

The initial condition check (16) is  $0 \leq 2(b+1) - 1$ , i.e. it is always satisfied. The two solutions of the constraint equation system are  $(\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}) = (1, 0, 2)$  and  $(\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}) = (0, 2, 0)$  and the relative controller structures are:

$$\begin{aligned} \mathbf{L}'_1 &= [2 \ 1 \ 2 \ 0 \ 0], \quad k'_1 = 2(b+1) - 1 \\ \mathbf{D}_{c1} &= -\mathbf{L}'_1\mathbf{D}_p = [2 \ 0 \ 0 \ -1 \ -2] \\ \mu_{c01} &= k'_1 - \mathbf{L}'_1^{-1}p_0 = k'_1 \\ \mathbf{L}'_2 &= [1 \ 0 \ 1 \ 0 \ 0], \quad k'_2 = k \\ \mathbf{D}_{c2} &= -\mathbf{L}'_2\mathbf{D}_p = [1 \ 0 \ 0 \ 0 \ -1] \\ \mu_{c0} &= k'_2 - \mathbf{L}'_2^{-1}p_0 = b \end{aligned}$$

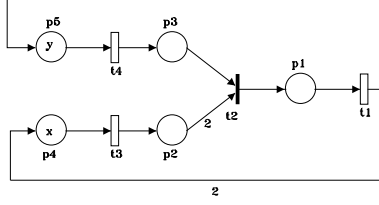


Figure 2: System in example 2.

Note that  $\mathbf{L}'_2 \leq \mathbf{L}'_1$ ,  $k_2 \leq k_1$ , i.e. the first solution isn't minimal: this is why we have to start the algorithm from the positive element of  $\mathbf{N}(r, s)$  such that  $b_s$  is minimum.

#### 4 SUBOPTIMALITY CRITERIA

Every solution  $(\alpha_{1,1}^*, \dots, \alpha_{x, m_{uc}}^*)$  of the previous algorithm is suboptimal, i.e. it decreases the cardinality of  $\mathcal{M}_c$  more than it is necessary.

If the state space is finite, then a very little efficient method is the direct computation of the reachability set of each monitor controlled net in order to choose the solution with the larger reachability set. This computation may be computation demanding for complex nets, so the cardinality of reachability set is not an efficient suboptimality index.

On the other hand an heuristic measure can be introduced, that in the next examples will be shown to give similar results to the previous criterion. Because each transition firing implies a marking update, a different criterion is to compute in the closed loop net the maximum value of transition firings allowed by each monitor structure *before monitor places become empty* and to choose the monitor structure whose value is greater.

We proceed as follows:

- a) we add the monitor structure to the open loop net;
- b) we remove the arcs inputing to monitor places;
- c) we solve the following ILP problem:

$$\begin{aligned}
 & \max_{\boldsymbol{\alpha}, \bar{\cdot}, \bar{\cdot}_c} \sum_{i=1}^n \sigma_i \\
 & \text{s.t.} \begin{cases} \bar{\cdot} - \mathbf{D}\boldsymbol{\alpha} = \bar{\cdot}_0 \\ \bar{\cdot}_c - \mathbf{D}_c^- \boldsymbol{\alpha} = \bar{\cdot}_{c0} \\ \boldsymbol{\alpha}, \bar{\cdot}, \bar{\cdot}_c \geq \mathbf{0} \end{cases} \quad (17)
 \end{aligned}$$

with  $\mathbf{D}_c^-$  is the  $\mathbf{D}_c$  part containing only the arcs from monitor places to transitions;

- d) we denote the max value of objective function as C index.

The ILP problem above will have a finite solution only if all the infinite sequence of transitions of the net contain every transition infinite number of times, so that all the transitions are disabled when the monitor places are empty.

#### Example 2

x	y	$R_1$	$R_2$	$C_1$	$C_2$	x	y	$R_1$	$R_2$	$C_1$	$C_2$
3	6	<b>40</b>	10	<b>12</b>	8	6	3	<b>22</b>	19	9	<b>11</b>
3	9	<b>58</b>	10	<b>15</b>	8	9	3	<b>22</b>	<b>28</b>	9	<b>14</b>
3	12	<b>76</b>	10	<b>18</b>	8	12	3	<b>22</b>	<b>37</b>	9	<b>17</b>
3	15	<b>94</b>	10	<b>21</b>	8	15	3	<b>22</b>	<b>46</b>	9	<b>22</b>
4	8	<b>52</b>	13	<b>14</b>	9	8	4	<b>28</b>	25	10	<b>13</b>
4	12	<b>76</b>	13	<b>18</b>	9	12	4	<b>28</b>	<b>37</b>	10	<b>17</b>
4	16	<b>100</b>	13	<b>26</b>	9	16	4	<b>28</b>	<b>49</b>	10	<b>21</b>
4	20	<b>124</b>	13	<b>21</b>	9	20	4	<b>28</b>	<b>61</b>	10	<b>25</b>
5	10	<b>64</b>	16	<b>16</b>	10	10	5	<b>34</b>	31	11	<b>10</b>
5	15	<b>94</b>	16	<b>21</b>	10	15	5	<b>34</b>	<b>46</b>	11	<b>22</b>
5	20	<b>124</b>	16	<b>26</b>	10	20	5	<b>34</b>	<b>61</b>	11	<b>25</b>
5	25	<b>154</b>	16	<b>31</b>	10	25	5	<b>34</b>	<b>76</b>	11	<b>30</b>

Table 1: Cardinality of reachability set and C index varying the initial marking and the control net for the net in fig. 2

The incidence matrix of the net in Fig. 2 is

$$\mathbf{D}_p = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

The initial marking is  ${}^{-p_0} = [0 \ 0 \ 0 \ x \ y]$ . The control goal is  $\mu_1 \leq 1$ , i.e.  $\mathbf{L} = [1 \ 0 \ 0 \ 0 \ 0]$ ,  $k = 1$ . The constraint transformation problem solutions are

$$\mathbf{R}_1 = [0 \ \alpha_{1,2} \ \alpha_{1,3} \ 0 \ 0]$$

$$\mathbf{R}_2 = [2]$$

$$s.t. \ 2\alpha_{1,2} + \alpha_{1,3} = 2$$

The initial condition check (16) is  $0 \leq 3$ , i.e. it is always satisfied. The two solutions of the constraint equation system are  $(\alpha_{1,2}, \alpha_{1,3}) = (1, 0)$  and  $(\alpha_{1,2}, \alpha_{1,3}) = (0, 2)$  and the relative controller structures are:

$$\mathbf{L}'_1 = [2 \ 1 \ 0 \ 0 \ 0], \ k'_1 = 3$$

$$\mathbf{D}_{c1} = -\mathbf{L}'_1 \mathbf{D}_p = [2 \ 0 \ -1 \ 0]$$

$$\mu_{c01} = k'_1 - \mathbf{L}'_1 {}^{-p_0} = 3$$

$$\mathbf{L}'_2 = [1 \ 0 \ 1 \ 0 \ 0], \ k'_2 = 1$$

$$\mathbf{D}_{c2} = -\mathbf{L}'_2 \mathbf{D}_p = [1 \ 0 \ 0 \ -1]$$

$$\mu_{c02} = k'_2 - \mathbf{L}'_2 {}^{-p_0} = 1$$

Solving the system (17) we have that  $\max \sum_{i=1}^n \sigma_i = 6 + y$  for the net controlled by the first monitor structure and  $\max \sum_{i=1}^n \sigma_i = 5 + x$  for the net controlled by the second one. In table 1 are shown for each initial marking the cardinality of reachability set denoted as R and the C

index for the closed loop net in the case of first control structure  $(R_1, C_1)$  and the second one  $(R_2, C_2)$ . In bold character are denoted the value of the suboptimal structure for each criterion. Note that the results of two methods are similar and according to the C index if  $y$  is greater than  $x$  the first structure is the suboptimal one, otherwise the second structure is the suboptimal one. This is not true only for small value of  $x$  and  $y$ .

### Example 3

The incidence matrix of the net in Fig. 3 is

$$\mathbf{D}_p = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The initial marking is  ${}^{-p_0} = [0 \ 0 \ 0 \ 0 \ x \ y]$ . The control goal is  $\mu_1 \leq 1$ , i.e.  $\mathbf{L} = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$ ,  $k = 1$ . The two controller structures are:

$$\begin{aligned} \mathbf{L}'_1 &= [1 \ 1 \ 0 \ 1 \ 0 \ 0], k'_1 = 1 \\ \mathbf{D}_{c1} &= -\mathbf{L}'_1 \mathbf{D}_p = [1 \ 0 \ 0 \ 0 \ -1] \\ \mu_{c01} &= k'_1 - \mathbf{L}'_1 {}^{-p_0} = 1 \\ \mathbf{L}'_2 &= [1 \ 0 \ 1 \ 0 \ 0 \ 0], k'_2 = 1 \\ \mathbf{D}_{c2} &= -\mathbf{L}'_2 \mathbf{D}_p = [1 \ 0 \ 0 \ -1 \ 0] \\ \mu_{c02} &= k'_2 - \mathbf{L}'_2 {}^{-p_0} = 1 \end{aligned}$$

Solving the system (17) we have that  $\max \sum_{i=1}^n \sigma_i = 5 + y$  for the net controlled by the first monitor structure and  $\max \sum_{i=1}^n \sigma_i = 5 + 2x$  for the net controlled by the second one. In table 2 are shown for each initial marking in the range  $(x, y) = [1 : 6; 1 : 6]$  the cardinality of reachability set denoted as R and the C index for the closed loop net in the case of first control structure  $(R_1, C_1)$  and the second one  $(R_2, C_2)$ . Note that the results of two methods are similar and according to the C index if  $y$  is greater than  $x$  the first structure is the suboptimal one, otherwise the second structure is the suboptimal one as in the previous example. This is not true only for small value of  $x$  and  $y$ . In addition note that for the net controlled by the first control structure both the indexes of the two criteria are not depending on  $x$  and that for the second net they are not depending on  $y$ . Finally note that both the indexes depends on  $x$  value more than from  $y$  value.

## 5 CONCLUSIONS

This paper has presented an algorithm to obtain all monitors forcing a given set of generalized mutual exclusion constraints in compact form. It yields a parameterization in form of a unique incidence matrix depending on the value of parameters subject to a integer linear equation system. Because there is not a supremal element in this class of control net structures a suboptimality criterion based on the cardinality of reachability space state has been proposed. A less computation demanding heuristic criterion based on transition firings has been also given.

x	y	$R_1$	$C_1$	x	y	$R_2$	$C_2$
$1 \leq 6$	1	7	6	1	$1 \leq 6$	7	7
$1 \leq 6$	2	11	7	2	$1 \leq 6$	15	9
$1 \leq 6$	3	15	8	3	$1 \leq 6$	26	11
$1 \leq 6$	4	19	9	4	$1 \leq 6$	40	13
$1 \leq 6$	5	23	10	5	$1 \leq 6$	57	15
$1 \leq 6$	6	27	11	6	$1 \leq 6$	77	17

Table 2: Cardinality of reachability set and C index varying the initial marking and control net for the net in fig. 3

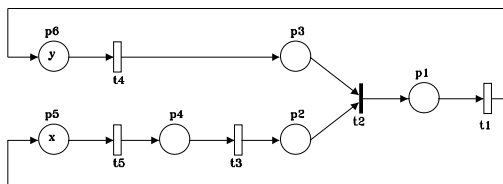


Figure 3: System in example 3.

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