A Spectral Rotation Approach for the Efficient Calculation of the Mutual Coupling Between Rectangular Apertures

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Abstract—In this letter, a procedure based on the spectral rotation is introduced that allows us to calculate the mutual coupling between apertures in the spectral domain without evaluating any convolution products (reaction integral) directly. A numerical evaluation of the computational cost is presented to show the efficiency of the method for electrically large problems. Numerical examples are given to highlight the agreement between the spatial and spectral rotation methods. Finally, the procedure is applied for the design of a horn antenna array with rotated apertures.

Index Terms—Horn antennas arrays, mutual coupling, numerical methods, spectral domain.

I. INTRODUCTION

RECTANGULAR aperture array antennas are common in many applications. In this context, a major step entails the computation of the near-field mutual admittances, which leads to convolution-based reaction integrals in the spatial domain [1]. Typically, the computation of the latter integrals places a heavy burden on the CPU time as well as memory requirements. A closer look reveals that the computational effort is very large for closely spaced apertures and significantly smaller for distant ones. In addition, the mutual coupling is inversely proportional to the aperture spacing, so the overall accuracy depends essentially on the coupling between closely spaced elements. This significantly increases the total computational load since both high accuracy and longer computational time are required for closely spaced elements [2].

To overcome this difficulty, spectral domain techniques have been widely investigated. Specifically, the 2-D spatial convolution in the spatial domain is transformed into a product in the 2-D spectral domain. However, spectral domain techniques have been used only for limited specific geometries, i.e., if the apertures are coplanar [3] or if they are part of cylindrical or spherical arrays [4], [5]. In this letter, we show how the mutual coupling between apertures can be efficiently computed by using a 2-D spectral domain technique, namely the spectral rotation approach. The spectral rotation approach has been introduced in [6], where it has been used for the calculation of the field over a generic plane. Here instead, the spectral rotation approach is used for calculating the coupling integral between arbitrarily oriented and non-coplanar apertures. Moreover, it is shown that its computational cost remains fixed at the increasing of the frequency; this is a very attractive feature when dealing with electrically large problems. Finally, the procedure is applied for the design of a horn antenna array with rotated apertures.

II. MUTUAL COUPLING

A. Formulation in Spatial and Spectral Domain

Let us consider the mutual coupling between two rectangular apertures radiating onto free space. We assume initially that the apertures are coplanar (Fig. 1); \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) represent the electric field modes on both apertures. Therefore, the equivalent magnetic currents are given by \( \mathbf{m}_1 = -2k_i z \times \mathbf{E}_i, \) \( i = 1, 2 \), and the coupling integral (CI) is

\[
\text{CI} = \iint_{S_2} \mathbf{H}_{\mathbf{m}_1} \cdot \mathbf{m}_{2d} dS.
\]

(1)

In (1), \( \mathbf{H}_{\mathbf{m}_1} \) represent the magnetic field generated by the magnetic current \( \mathbf{m}_1 \); note that the integral is extended on the surface limited by aperture #2. \( \mathbf{H}_{\mathbf{m}_1} \) can be expressed by using the Fitzgerald potential and (1) becomes

\[
\text{CI} = \iint_{S_2} j\omega \left[ \mathbf{P}_{\mathbf{m}_1} + \frac{1}{k_i^2} \nabla \nabla \mathbf{F}_{\mathbf{m}_1} \right] \cdot \mathbf{m}_{2d} dS.
\]

(2)
Since the determination of $\tilde{P}_{m_1}$ requires the calculation of a convolution product in the spatial domain, it is convenient to express (2) in the spectral domain ($k_x, k_y$). In particular, by using the procedure given in [7], we have
\[
CI = \frac{j \omega \varepsilon}{4\pi^2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{P}_{m_1}(k_x, k_y) \cdot \tilde{m}_2(-k_x, -k_y) dk_x dk_y + \frac{1}{k_z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T \left\{ \nabla \tilde{P}_{m_1} \right\}(k_x, k_y) \cdot T \left\{ \nabla \tilde{m}_2 \right\}(-k_x, -k_y) dk_x dk_y \right)
\]
where $k$ is the wavenumber. By denoting with $g$ the Green’s function, in the spectral domain it holds
\[
\tilde{P}_{m_1} = \tilde{g} \cdot \tilde{m}_1
\]
where
\[
\tilde{g} = T\{g\} = \frac{j}{k_z} = \frac{j}{\sqrt{k^2 - (k_x^2 + k_y^2)}}.
\]
Thus, (3) becomes
\[
CI = \frac{j \omega \varepsilon}{4\pi^2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{g} \cdot \tilde{m}_1 \cdot \tilde{m}_2(-k_x, -k_y) dk_x dk_y + \frac{1}{k_z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T \left\{ \nabla \tilde{P}_{m_1} \right\}(k_x, k_y) \cdot T \left\{ \nabla \tilde{m}_2 \right\}(-k_x, -k_y) dk_x dk_y \right)
\]
which can be written as
\[
CI = \frac{\omega \varepsilon}{4\pi^2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k_z} T \left\{ \nabla \tilde{m}_1 \right\}(k_x, k_y) \cdot T \left\{ \nabla \tilde{m}_2 \right\}(-k_x, -k_y) dk_x dk_y \right)
\]
\[
+ \frac{1}{k_z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k_z} T \left\{ \nabla \tilde{m}_1 \right\}(k_x, k_y) \cdot \tilde{m}_2(-k_x, -k_y) dk_x dk_y.
\]
\[
(7)
\]
B. Alternative Formulation: Spectral Rotation Approach
Let us now consider again (1), where $\tilde{H}_{m_1}$ represent the magnetic field generated by $m_1$. $\tilde{H}_{m_1}$ can be expressed by using the following compact notation:
\[
\tilde{H}_{m_1} = \frac{\omega \varepsilon}{G} \otimes \tilde{m}_1
\]
where
\[
\frac{\omega \varepsilon}{G} = j \omega \varepsilon \left( \tilde{I} - \frac{1}{k^2} \nabla \nabla \right)
\]
is the dyadic Green’s function pertaining to the magnetic field and $\tilde{I}$ is the identity matrix. By introducing a reference system $(x_2, y_2)$ on the aperture #2, we can write the spectrum of the magnetic field in the spectral domain $(k_{x_2}, k_{y_2})$ by resorting to the spectral rotation [6] as
\[
\tilde{H}_{m_1}^{\text{rot}} = \frac{\omega \varepsilon}{G} \cdot \tilde{m}_1^{\text{rot}}.
\]
\[
\tilde{m}_1^{\text{rot}}
\]
is the spectrum of $m_1$ expressed in the spectral domain $(k_{x_2}, k_{y_2})$, and it can be determined analytically since $\tilde{m}_1$ is analytically transformable. Thus, (1) can be written in the spectral domain $(k_{x_2}, k_{y_2})$ as
\[
CI = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\omega \varepsilon}{G} \cdot \tilde{m}_1^{\text{rot}} \cdot \tilde{m}_2 dk_{x_2} dk_{y_2}.
\]
\[
(11)
\]
It is worthwhile pointing out that (11) can be seen as a generalization of (7), and it can be used also when dealing with arbitrarily oriented and noncoplanar apertures (Fig. 2). The integrand in (11) can be expressed in a closed form. A closer look reveals that a theta-rotation between the reference systems of two apertures [Fig. 3(a)] can be handled by applying the following analytic continuation procedure:
\[
\tilde{m}_1^{\text{rot}}(k_{x_2}, k_{y_2}) = \tilde{m}_1(k_{x}, k_{y}) \bigg|_{k_x = k_{x_2} \cos \theta - k_{y_2} \sin \theta}^{k_y = k_{y_2} \sin \theta + k_{x_2} \cos \theta}
\]
\[
(12)
\]
where $k_{x_2}$ is obtained from the relation $k_{x_2}^2 + k_{y_2}^2 + k_{z_2}^2 = k^2$. For a phi-rotation [Fig. 3(b)] we have
\[
\tilde{m}_1^{\text{rot}}(k_{x_2}, k_{y_2}) = \tilde{m}_1(k_{x}, k_{y}) \bigg|_{k_x = k_{x_2} \cos \phi - k_{y_2} \sin \phi}^{k_y = k_{y_2} \sin \phi + k_{x_2} \cos \phi}.
\]
\[
(13)
\]
By combining translations and rotations around theta and phi, arbitrarily oriented and noncoplanar apertures can be handled. The integral (11) must be numerically evaluated: In this context, care must be exercised to avoid the sampling at the singularities of the transform of the dyadic Green’s function [8]. For distant apertures, the spectral rotation integrand can be highly oscillatory, but the reduced accuracy needed in this coupling computation allows using the weighted average algorithm [9]. A comparison of the computational costs to be paid for calculating the reaction integral through the expression in (11) or through...
that in (1), i.e., either in the spectral or in the space domain, is provided in Section III. It is demonstrated that the calculation in the spectral domain is more efficient for electrically large problems. In the following, the two procedures will be briefly referred to as spectral and spatial approaches, respectively.

III. NUMERICAL RESULTS

A numerical example will be presented in order to compare the computational cost of the spectral and the spatial approaches. Specifically, we consider two apertures displaced as shown in Fig. 4(a), i.e., with the $(x-x_2)$-axis rotated by 15° (rotation on phi); we assume a TE$_{10}$ mode aperture field distribution. The dimensions of the apertures are $a = 90$ mm and $b = 60$ mm. The mutual coupling is calculated by (1) in the spatial domain and (11) in the spectral domain.

Conventional approach in spatial domain resorts to Rao–Wilton–Glisson (RWG) discretization [10]. In particular, RWG basis functions with a step equal to $\Delta x = \Delta y = \lambda/15$ have been used for evaluating (1). In this case, the computational cost depends on the number of RWG basis functions, i.e., on the frequency. In our test, the frequency is assumed to vary from 5 to 10 GHz. Let $N_x = a/\Delta x$ and $N_y = b/\Delta y$ be the number of spatial samples, respectively, along the $x$- and $y$-directions. The number of basis functions on each aperture is

$$N_B = (N_x + 1) \times (N_y + 1) + N_x N_y - 2 \times N_x - 2 \times N_y.$$  \hspace{1cm} (14)

Thus, the computational cost of (1) is $O(N_B^2)$. The cost behavior in terms of the number of operations is shown in Fig. 4(b) (dashed curve) as a function of the frequency.

On the other hand, turning to the spectral domain, the integrand in (11) can be expressed in a closed form since $\tilde{m}_1$ and $\tilde{m}_2$ are analytically transformable. The integration domain is unbounded, but it can be suitably truncated since the integrand decreases very rapidly. Integral (11) must be evaluated numerically, and the discretization step $\Delta k_{x_2} = \Delta k_{y_2}$ must be suitably chosen in order to guarantee the accuracy of the results. The functions $\tilde{m}_1$ and $\tilde{m}_2$ become steeper when the frequency, i.e., the electrical dimension of the apertures, increases. As a consequence, a smaller discretization step $\Delta k_{x_2} = \Delta k_{y_2}$ is required. However, the integrand decreases more rapidly for increasing frequencies, so the integration domain can be truncated at a lower value. The result is that the number $N_{k_x}, N_{k_y}$ of the spectral samples is virtually independent of the frequency. As a consequence, the computational cost remains fixed to $O(N_{k_x} \times N_{k_y}) = O(N_B^2)$. For the example under test, a discretization step $\Delta k_{x_2} = \Delta k_{y_2} = k/100$ (k being the free-space wavenumber) and a number of samples $N_{k_x} = N_{k_y} = 1000$ have been chosen to guarantee a relative error with respect to spatial approach lower than 1% for all the frequencies.

As apparent in Fig. 4(b), the computational cost of the spectral approach (continuous line) is constant for increasing values of the frequency, i.e., for increasing values of the electrical dimension of the apertures, providing more efficiency than the spatial approach for electrically large problems. The same behavior has been obtained when assuming the apertures having a rotation on theta. Moreover, it is worthwhile to point out that a nonuniform sampling can be used to further reduce the computational burden of the spectral approach. This can be achieved by using a fine spectral sampling in the regions near the singularities of the transform of the dyadic Green’s function, while employing a coarser spectral sampling in regions far from the same singularities.

Before proceeding, it is worthwhile pointing out the following.

1) The number of basis functions $N_B$ of (14) has been obtained by resorting to RWG bases; a different choice of basis functions, i.e., roof top functions defined on a regular grid, will lead to a different $N_B$.

2) No entire domain modes approach is used for evaluating (11). Specifically, integral (11) is evaluated numerically through a discretization over a regular grid carried out in the spectral domain.

3) If the analytical transforms of the modes are not available, a discrete Fourier transform/fast Fourier transform (DFT/FFT) procedure can be used for calculating $\tilde{m}_1$ and $\tilde{m}_2$. In this latter case, together with the computational cost for evaluating integral (11), the additional cost for calculating $\tilde{m}_1$ and $\tilde{m}_2$ should be also considered.

Numerical examples are now given to highlight the agreement between the spatial and spectral rotation methods. We consider two square apertures, and we assume TE$_{10}$ mode aperture field distributions. The dimension of the apertures is set to 0.1 m. First, the case of two apertures lying on the same plane with the $(x-x_2)$-axis rotated by 15° is considered (rotation on phi). The top part of Fig. 5 shows the magnitude of the mutual coupling calculated when considering separation distance between the aperture centers of 0.2 m for the frequency range of 3–6 GHz. The continuous line refers to the spatial approach that resorts to a $\lambda/10$ RWG discretization of the modes; the dotted line refers to the spectral approach. The bottom part of Fig. 5 refers to the case of the two apertures having the $(z-z_2)$-axis rotated by 90° (rotation on theta).

Finally, the procedure is applied for the design of a horn antenna array with rotated apertures. Specifically, a pyramidal horn antenna array comprising two elements is considered as a testing example. The details of the radiating element are given in Fig. 6, while Fig. 7(a) shows a planar view of the array. The analysis can be decomposed into two separate steps, i.e., the analysis of the waveguide transitions from the feeding section to the radiating apertures, and the analysis of the radiating apertures

![Fig. 4. Comparison of the computational costs in terms of number of multiplications. (a) Geometry of the problem. (b) Number of multiplications in spatial and spectral domain. Log-scale is used for the $y$-axis.](image)
Fig. 5. Coupling between TE\textsubscript{10} mode aperture field distributions. (top) \((x - x_2)\)-axis rotated by 15\degree. (bottom) \((z - z_2)\)-axis rotated by 90\degree. A log-scale is used for the \(y\)-axis.

Fig. 6. Pyramidal horn antenna.

excited by a complete modal distribution (see [11] for more details). In this context, Fig. 7(b) shows the magnitude of the coupling \((S_{12})\) within a frequency range of 5–10 GHz calculated when employing the spectral domain approach (black line) and selecting a total number of 25 TE and 18 TM modes over the radiating aperture. In the same figure, the dashed line refers to a conventional spatial domain approach (\(\lambda/15\) RWG discretization). For this last example, a numerical evaluation of the computational time has been performed to highlight the efficiency of the spectral domain approach. Specifically, the spatial approach requires 45 s (Intel Pentium M 2 GHz with 1 GB RAM), while the computational time decreases to 14 s when the spectral approach is employed.

IV. CONCLUSION

An approach based on the spectral rotation for the computation of the coupling between apertures has been presented in this communication. The technique reveals computational efficiency and permits addressing more general problems involving arbitrarily oriented and non-coplanar apertures, being suited for conformal array analysis.

Numerical examples have been presented to highlight the agreement between the spatial and spectral rotation methods. The proposed approach is well suited for solving problems involving analytically transformable modes. However, if the analytical transforms of the modes are not available, a DFT/FFT procedure can be used. It follows that problems involving non-conventional-shaped horn antennas can be addressed as well.

REFERENCES