DECENTRALIZED CONTROL OF OPEN-CHANNEL HYDRAULIC SYSTEMS

Author: Carla Seatzu

Supervisor: Prof. Giampaolo Usai

December 1999
Contents

Introduction 4

1 Regulation of open–channels: a classification of the existing procedures 7
  1.1 Introduction .......................................................... 7
  1.2 Considered variables ............................................... 7
  1.3 Logics of control .................................................... 12
  1.4 Methods of design .................................................. 15
  1.5 Field implementation ............................................... 16
  1.6 Conclusions ......................................................... 18

2 Dynamical models of open channels 20
  2.1 Introduction .......................................................... 20
  2.2 State of art ............................................................ 20
  2.3 Dynamical models for the synthesis of the controllers ............. 23
    2.3.1 Reference model ............................................... 24
    2.3.2 Approximate nominal model ................................... 33
  2.4 A linear decoupled model for the synthesis of a decentralized observer . . 36
  2.5 SIC, a simulation model for canal automation design ..................... 38
  2.6 Conclusions ......................................................... 39

3 Decentralized control laws via parameter optimization 40
3.1 Introduction ................................................................. 40
3.2 Decentralized control by solving a linear least squares problem ........ 42
  3.2.1 Design procedure .................................................. 42
  3.2.2 A numerical application ......................................... 45
3.3 Decentralized control via Edmunds’ algorithm .......................... 49
  3.3.1 Edmunds’ algorithm ................................................. 49
  3.3.2 Decentralized control law ......................................... 53
  3.3.3 PI decentralized control law .................................... 53
  3.3.4 A numerical application ......................................... 54
3.4 Decentralized control via $H_2$ norm minimization ....................... 59
  3.4.1 Conditions on $H_2$ norm ......................................... 59
  3.4.2 Decentralized control law ......................................... 61
  3.4.3 PI decentralized control law .................................... 63
  3.4.4 A numerical application ......................................... 63
3.5 Other methodologies .......................................................... 67
  3.5.1 The method of inequalities .................................... 67
  3.5.2 Multi–objective optimization .................................... 70
3.6 Conclusions ................................................................. 71

4 Decentralized control laws via eigenstructure assignment .......... 73
  4.1 Introduction ................................................................. 73
  4.2 Design procedure .......................................................... 74
    4.2.1 Eigenvalue assignment ......................................... 75
    4.2.2 Eigenstructure assignment ..................................... 81
  4.3 A numerical application ................................................. 81
  4.4 Conclusions ................................................................. 90

5 A decentralized volume variation observer .......................... 93
5.1 Introduction ................................................................. 93
5.2 Background on asymptotic state observers ........................................ 94
  5.2.1 Method I ............................................................ 95
  5.2.2 Method II .......................................................... 96
5.3 A numerical application ....................................................... 97
  5.3.1 Simulation 1 ......................................................... 99
  5.3.2 Simulation 2 ......................................................... 99
  5.3.3 Simulation 3 ......................................................... 101
5.4 Conclusions ................................................................. 102

6 Stability robustness analysis ................................................. 103
  6.1 Introduction ............................................................. 103
  6.2 Unstructured uncertainty .................................................... 104
  6.3 Stability robustness ......................................................... 105
  6.4 Numerical applications ....................................................... 106
    6.4.1 Robustness analysis with no state observer ....................... 106
    6.4.2 Robustness analysis with asymptotic state observer ............. 110
  6.5 Conclusions ............................................................. 115

Conclusions ................................................................. 116

Bibliography ................................................................. 118
Introduction

Many open–channels regulation methods have been developed in the world. These methods differ from a country or region to another. They range from the simplest methods, developed more than 2000 years ago, to the most sophisticated ones developed recently, or under development. Furthermore, a wide variety of mathematical models describing the dynamic behaviour of open–channels have been proposed in the literature. Sometimes differences among them are really significant and involve the use of completely different control techniques for the synthesis of the controller.

After a careful examination of the literature on the subject, it has been evaluated that a valid solution to the modelling problem has been proposed by Corriga et al. in [19, 21] where two models deduced from the Saint-Venant equations have been formulated. The first one, denoted as the reference model, expresses the dynamic relationships, in terms of transcendental functions, between the gate opening sections and the corresponding stored volume variations in the different canal reaches with respect to a reference configuration of uniform flow. The second one, denoted as approximate nominal model, is obtained from the previous one by means of a series expansion around $s = 0$, where the most significant phenomena take place. It is a state variable linear and time invariant model. The state variables and the control variables are equal to the output and input variables of the preceding model respectively. The second model enables the use of classical control techniques in the time domain and can thus be easily used for the solution of synthesis problems. It seemed to be a good trade-off between accuracy and simplicity. In fact, as far as the author knows, it is the smallest size linear state space model, being its order equal to the number of pools in the canal. This model enables us to design effective constant volume control laws. Note that the choice of volume variations as controlled variables arises from the requirement to design controllers less sensitive to perturbations, and to reduce the transient time, in particular to reduce it to the time necessary to the settling of new flow rate values in each canal reach.

As it is well known, when considering large scale systems, like open-channel hydraulic networks, decentralized decision making is needed. Thus, the composite system has several local stations, of which, at each station, the controller observes only local system outputs and controls only local inputs; all the controllers, however, are involved in controlling such large composite systems. The reason why this decentralized constraint usually arises in controlling
such large composite systems is due to the fact that a centralized controller, i.e., a single controller which observes all outputs of the system to control all inputs of the system, usually will require excessive information-gathering to make such a controller practical to apply. Thus, the study of the control of large scale systems directly leads to the study of decentralized control.

The problem of designing a decentralized control law has been solved by the author with different design methodologies. In particular, it has been firstly seen as a problem of parameter optimization, so as to impose the desired structure to the feedback controller [84, 85, 86, 88, 89, 90]. In all cited works satisfactory results have been obtained. Note that all simulations have been carried out on a completely nonlinear model developed at Cemagref [56].

In [88] a really simple procedure in the time-domain has been proposed and it has been applied to the actual case. Such a procedure, even if it presents some drawbacks with respect to much more sophisticated techniques, has the considerable advantage that its computational complexity weakly increases with the dimension of the problem, thus making it interesting for the case at hand. It consists in the evaluation of the decentralized feedback gains such that the closed-loop behaviour of the decentralized system approaches, in the least squares sense, the state evolution of a target centralized closed-loop plant. In such a way, the synthesis problem reduces to the problem of identifying a set of unknown parameters in a linear system of ordinary differential equations. Since also the time derivatives of the target state can be computed, such a problem can be solved as a simple linear least squares problem, where no difficulty due to local minima and good initial estimate occurs.

In [86, 90] a specific frequency-domain optimization problem has been solved to design both a proportional and a proportional integral decentralized controller. The algorithm that involves the solution is rather crude. Nevertheless, when used intelligently it is capable of finding very good controllers of relatively low complexity. Its complexity is comparable to that of the above algorithm in the time-domain. However, it presents the advantage that its range of applicability is wider. In fact, it enables us to deal with systems whose dynamics is modeled by non-rational transfer function matrices. In particular, in the case at hand, it enabled us to use the more general reference model.

Finally, decentralized controllers have been designed by minimizing the $H_2$ norm of an appropriate transfer function matrix. In particular both a proportional and a proportional integral decentralized control laws have been proposed [84, 85, 89]. However, in such a case a non-convex optimization problem has to be solved, thus problems related to local minima may occur. Nevertheless, this method has revealed to be really effective when applied to the actual problem.

A completely different approach to the problem of designing decentralized feedback controllers consists in the eigenstructure assignment (simultaneous assignment of eigenvalues and eigenvectors) and it has been applied to the system of interest in [93, 94]. The eigenstructure of a linear multivariable system governs its dynamic behaviour: the speed of the dynamic response
of a linear system depends on its eigenvalues whereas the "relative shape" of the dynamic response depends on the associated eigenvectors. A lot of solutions have been proposed in the literature on this subject and it seems that a valid one may be that firstly proposed by Lu et al. in [52]. A parametric expression for the set of feedback gains of decentralized controllers which achieve the desired eigenvalue assignments can be obtained. The free parameters in this parametric expression can then be used to assign eigenvalues and eigenvectors simultaneously.

It is important to note that all the above decentralized control laws require the measurement of the volume variations in all canal reaches. For this reason, the design of a volume variations observer has also been proposed [93]. In particular, the main purpose is that of obtaining a good estimation of all state variables even in presence of unknown disturbances and errors in the initial volume estimate. The main idea of our work is that of reconstructing each state variable with the only knowledge of the opening variations of the two gates delimiting the reach whose volume variation is to be detected, and their upstream and downstream level variations. Therefore, the proposed observer reconstructs each state variable on the base of only local measurements, thus keeping the advantage of decentralization in the control.

Since the fundamental property which must be retained for all possible perturbations of the plant is stability of the feedback system, we have also studied whether the proposed feedback designs are robust when output-multiplicative perturbations occur due to the low-frequency approximations and to the variations with respect to the reference configuration of uniform flow. Note that stability has been proved also in presence of the state observer.
Chapter 1

Regulation of open–channels: a classification of the existing procedures

1.1 Introduction

Many irrigation canal regulation methods have been developed in the world. These methods differ from a country or region to another. They range from the simplest methods, developed more than 2000 years ago, to the most sophisticated ones developed recently, or under development.

In this chapter we make a state of art of the problem under examination and we provide a detailed characterization of irrigation canal regulation methods, based on four criteria: considered variables (controlled, measured, and control action variables), logic of control, design method (structure and technique) and field implementation (configuration and device). This classification has been firstly proposed by Malaterre in [54, 55]. Many other authors have attempted to classify the different canal regulation methods developed in the world [1, 2, 3, 13, 15, 25, 108]. Generally, these authors are not control engineers, but irrigation, hydraulic or civil engineers. The concepts they use to differentiate canal regulation methods are from different points of view (e.g.: upstream control and remote control). Furthermore, their definitions are different and sometimes contradictory.

1.2 Considered variables

In control theory, a system is usually represented as in figure 1.1. U is the system control action variables (e.g.: gate openings, increments of gate opening, discharges), which are acted on, so
Controlled variables

Controlled variables are system variables to which targets are assigned. Controlled variables on an irrigation canal can be of two types: flows (discharges) or efforts (water levels), according to the terminology used in physical system modeling through bond–graphs. They can equally be volumes, which means integral of flows (discharges) in reference to time, or integral of water levels in reference to space.
Discharges

The needs of irrigation canal users are defined mainly in terms of discharge. For example, agricultural needs are expressed in terms of given discharges delivered to a plot, to a secondary canal, or to a pumping section, environmental needs as tailed discharge, or minimal discharge, urban needs as discharges delivered to a house or to a city water filtration plant, and industrial needs as discharges delivered to a factory. Natural or artificial storage reservoirs are sometimes available. Users’ needs can then be defined in a more flexible way, in terms of volume distributed over a time period. In this case, the controlled variable is no longer a given value of discharge, but a volume, which is the integral of a discharge over a given time period. Discharge fluctuations are then authorized, and neutralized by the capacity of the storage reservoirs. However, these reservoirs are expensive and of limited sizes, and constraints of distribution never suppress needs expressed in terms of discharge.

Consequently, all free surface hydraulic systems have to be managed, directly or indirectly, in order to satisfy users’ demands in discharge. Considering the nature of the physical phenomenon at stake (gravity open channel flow from upstream to downstream), these demands in discharge can be satisfied only from the source situated at the upstream end of the system, by draining the upstream reservoirs. Examples of regulation methods controlling discharges are [15, 67, 74, 75, 76, 77, 78].

Water levels

Contrary to discharges, water levels can be easily measured in free surface canals and rivers. Furthermore, constraints of feeding gravity turnouts, stability of canal banks, efforts to reduce weed growth, constitution of intermediate water storage volumes, risks of overflow, are expressed in terms of water levels. Controlled water levels ”$y$” can be upstream ($y_{up}$, figure 1.3.a), downstream ($y_{dn}$, figure 1.3.b), or intermediate inside the pool ($y_{in}$, figure 1.3.c). On figures 1.3.a and b, controlled water level values are equal at null and maximum discharges. Operational characteristics are very different depending on the location of ”$y$”.

One of the advantages of controlling upstream water levels is that a storage volume $V$ is available between the null discharge volume and the maximum discharge volume. It allows for
rapid response to unforeseen demands of turnouts or downstream reaches and for storing water in case of a consumption reduction. But canal banks have to be horizontal, which is expensive. AVIS and AVIO gates are examples of such methods.

When downstream water levels are controlled, canal banks can follow the field natural slope, which reduces construction costs. But, no storage volume is available between the null discharge volume and the maximum discharge volume. Therefore, the system cannot respond rapidly to unforeseen demands. The excess water cannot be stored locally and is "lost" in the downstream pools. Examples of downstream water level control are [12, 26, 108].

Controlling a particular intermediate water level, close to the middle of the pool, is equivalent to controlling the volume stored in the pool. This water level can be measured directly (no example has been found), or can be obtained as a linear combination of an upstream and a downstream water level (e.g.: BIVAL [15, 108]). Controlling an intermediate water level is a compromise between the two previous options, in terms of construction cost and availability in storage volume V. Indeed, banks have to be horizontal only downstream of the controlled intermediate water level. But, one or several distant water levels have to be measured, which implies the installation of transmission lines or of a measurement network. BIVAL [15, 108] is the only example of such a method.

**Volumes**

This method offers the advantage of drastically reducing transients by making the flow rate variations in each reach independent of the stored volumes. On the qualitative level, it is possible to say that transient time is reduced to the time necessary to the settling of the new flow rate values in each reach. These methods are applicable to irrigation canals with important storage volumes, and equipped with turnouts whose feeding is not dependent on water levels in the main canals (e.g.: pumping stations). Examples of such methods are [13, 18, 19, 84, 85, 86, 89, 90, 93, 94].

**Measured variables**

Measured variables on irrigation canals are generally water levels. In some case, measured variables can be discharges. A discharge can be measured with specific equipment (based in general on the measure of one or several flow velocities, an ultrasonic or elettromagnetic device), through a cross structure equation \( Q(z_1, z_2, w) \) (where \( z_1 \) and \( z_2 \) are the gate upstream and downstream water levels and \( w \) is the gate opening section), or a local control section rating curve \( Q(z) \) (where \( z \) is a water level) with a sufficient precision. Finally, measured variables can be volumes, evaluated by measuring several water levels along the canal, or by evaluating input–output discharge balance (e.g. Dynamic Regulation).
Figure 1.3: (a) Control of the upstream level of the pool; (b) control of the downstream level of the pool; (c) Control of an intermediate water level.
Control action variables

Control action variables U are generally either gate openings, increments of gate opening, discharges, or increments of discharge. Gate openings have the advantage of taking into account the complex dynamics linking this opening with the local discharge and upstream and downstream water levels. These dynamics are important and it can be hazardous not to take them into account. Considering discharge as the control action variable allows for decoupling of the different subsystems. This is interesting when monovariable controllers are used in series. However, the dynamics of the local controller linking the discharge (control action U) to the gate opening (elementary control action variable V) are not taken into account in the global controller. Therefore, the quality of the behaviour of the global controller cannot be assessed, since important dynamics are neglected in the design of the controller. If the control action variable U is a discharge, it is necessary to convert it into gate opening V, applicable to the system. This conversion can be done through the inversion of the device static equation \( Q(z_1, z_2, w) \), or by a local dynamic controller.

### 1.3 Logics of control

Control engineers clearly define the logic of control concept depending on the type of information used to calculate the control action variables of the system. It can be “closed–loop” or “open–loop”.

#### Closed–loop control

Closed–loops can be applied to all the controlled variables: discharges, water levels and volumes. Examples of closed–loops in discharge are [51, 67, 76].

Two types of closed–loops in water level are defined depending on the relative locations of the control action and controlled variable. Flows in irrigation canals are generally subcritical. Therefore, a water level can be controlled by modification of the downstream water level (if the controlled water level belongs to the corresponding backwater curve). These modifications are made, respectively, from an upstream structure (see figure 1.4.a), or a downstream structure (see figure 1.4.b). Paradoxically, for technological reasons, the first logic of control is called feedback downstream control, the second logic of control is called feedback upstream control.

Water level downstream control generates indirectly a discharge closed–loop control, since it is obtained from the modification of the upstream discharge. This characteristic is an essential property of the water level downstream control. Thanks to this remarkable particularity, a water level downstream control is sufficient, and no supplementary discharge control loop is necessary (neither in open–loop, nor in closed–loop). Examples of such methods are AVIS, AVIO gates, PID [15], PIR [26].
Figure 1.4: (a) Control of the upstream level of the pool; (b) control of the downstream level of the pool; (c) Control of an intermediate water level.
Contrary to water level downstream control, water level upstream control does not generate any discharge control, since it is obtained by simple modification of the downstream water level. Therefore, a water level upstream control is not sufficient. It has to be completed by an explicit discharge control loop (in open–loop or in closed–loop). Example of such methods are AMIL gates.

Some water level control methods combine upstream and downstream control logics (see figure 1.4.c). They are called feedback mixed control. Since they also indirectly generates a closed–loop discharge control, they are often called, to simplify, downstream controls. Examples of such methods are LQR [4, 5, 18, 19, 34, 35, 43, 54, 61, 62, 63, 64, 79].

The latter are sometimes designed as ”associated levels gates”, because, in certain hydraulic conditions, the purpose of the gate is to maintain a constant difference between its upstream and downstream water levels. We consider this gate as a ”mixed control” gate, because modification of a water level in a pool implies the combined reaction of the two gates located upstream and downstream of this water level.

In the case of a closed–loop in volume, system control structures are operated in function of errors between controlled volumes in various pools and the corresponding target volumes. In general, the volume of a pool is controlled through an upstream discharge modification. Therefore, the volume control generates implicitly a discharge control. Volumes could also be controlled through a downstream level modification.

For complex processes, like dead time processes (i.e., processes with time delays), closed–loop does not guarantee entire satisfaction. In the case of irrigation canals, time delays between upstream control actions and downstream controlled variables are important (a few minutes to several hours). A single closed–loop can function correctly only if important storage volumes are available. Indeed, control delay is at least, equal to the system delay. But storage volumes imply high construction costs. The quality of the control can be considerably improved by adding an open–loop [83].

Open–loop control

In open–loop, the control action U is calculated knowing the dynamics of the system (using a model), the target output and possibly an estimation of perturbations. The open–loop can compensate inherent system time delays by anticipating users’ needs. The needs have to be estimated as precisely as possible. They should take into account climatic, agronomic and sociological data, as well as recordings of the water consumption of previous weeks or seasons. An open–loop is generally insufficient, due to model errors, perturbation estimation errors, and unknown perturbations. Open–loops can be applied to all the controlled variables: discharge, water level and volume.

Examples of open–loops in discharge are given depending on the calculation method used:
by model inversion: kinematics wave or pure delay, diffusive wave [76], dynamic wave 
[15, 50]; these different methods are studied and compared by Chevereau [15];

• by optimization [50, 70, 99].

Examples of open–loops in water level are:

• by inversion of the dynamic wave model [33, 50];
• by optimization [50, 99];
• by simulation [6];
• some examples are quoted both in control of discharges and of water levels; for example, 
the dynamic wave model calculates simultaneously these two types of variables;
• an example of open–loop in volume is [70].

Therefore, we can conclude that both closed–loops and open–loops have advantages and 
limitations. For these reasons, the combination of open–loop and closed–loop is often used, the 
closed–loop allowing to compensate the open–loop errors.

1.4 Methods of design

In these section we cite different examples of irrigation canal regulation methods dividing them 
in two different sets: monovariable and multivariable methods.

Monovariable methods

Heuristic monovariable methods have been developed based on hydraulics and not on control 
theory [12, 108]. Although quoted in the literature they are hardly operational and too site spe-
cific. Monovariable methods require to split the overall system into several subsystems without 
explicitly taking into account interactions between them. An irrigation canal is a multivariable 
system presenting strong interactions between subsystems. For example, the operation of a 
gate influences several upstream and downstream pools. The decoupling technique has been 
applied to ELFLO controller [80]. It restrains, as far as possible, the influence of one control 
action on the unique regulator controlled output. The global monovariable process can then 
be considered as a series of independent monovariable non interactive processes evolving in 
parallel. This is possible if the number of inputs is superior or equal to the number of out-
puts. Performance of a controller can be greatly improved through decoupling. Decoupling 
requires a linear model of the system. Its performance is therefore decreased due to unknown 
perturbations and model errors [26, 80].
Multivariable methods

Control engineers have developed several multivariable methods. However, only recently they have been used on canals [84, 85, 86, 89, 90, 93, 94].

Different model inversion methods are described in the literature, leading generally to open–loop controllers [15, 50], and more rarely to closed–loop controllers [51].

Optimization methods have also been developed. These methods are, in essence, multivariable. Different methods exist: linear optimization [70], nonlinear optimization [50, 99], and LQR [4, 19, 34, 35, 43, 54, 63, 64, 79]. The classical nonlinear optimization leads solely to an open-loop, sensitive to errors and perturbations. In order to introduce a closed–loop, the optimization has to be processed periodically (for example at each time step). This complicates the method and limits its applications due to real–time constraints. Furthermore, the determination of real initial conditions, required for the optimization, is not easy. On the other hand, LQR methods, based on a state space realization, incorporate, in essence, an open–loop and a closed–loop.

The implementation of multivariable methods is far more complex than for monovariable methods.

1.5 Field implementation

This section presents different types of field implementation of canal regulation methods. They are briefly discussed hereafter. Field implementation must not be confused with design method.

Different aspects of field implementation of regulation methods can be distinguished. They are the configuration (e.g. localized, centralized), the devices (e.g. wires, manual gates, automatic gates), the instrumentation (e.g. water level sensor, discharge measurement device), communications (e.g. messenger, telephone line, radio–transmission), calculation and data processing (e.g. empirical method, abacus, hydromechanical principle, calculator, microcomputer, workstation).

The following subsections focus on the aspects of configuration and devices. The latter are the most important ones for the understanding of the characteristics of a canal regulation method, including its advantages, disadvantages, performance and constraints.

Configuration

Field implementation concerns the spatial links between controllers, devices and controlled variables, and hierarchical links between controllers. It can be classified according to the following partition rule: local close, local distant, semi–local, centralized and hierarchical.
Figure 1.5: (a) Local close configuration; (b) local distant configuration; (c) semi-local configuration; (d) centralized configuration; (e) hierarchical configuration.
In local closed configuration (figure 1.5.a) each device is controlled by an independent controller, using local information (measured variables), located close (around a few meters) to the device.

Local distant configurations (figure 1.5.b) are similar to local close configurations, except that measured variables are distant from the device (for example they can be located at the downstream end of the pool).

In semi–local configuration (figure 1.5.c) controllers are no longer independent. They explicitly use data generated by adjacent controllers. For example, a controller can use the control action variable of the next downstream controller.

In centralized configuration (figure 1.5.d) all control actions are generated by a central controller (human being or computer), distant from the different devices. Control action are gate openings, discharges, or water levels. Centralized control allows for supervision and remote control of the system. However, its implementation is more complex and more sensitive to hardware breakdowns than localized configurations.

In hierarchical configuration (figure 1.5.e) control actions depend on different hierarchical decision-making levels. For example, automatic local distant controllers communicate with a centralized controller. This latter is in charge of target modifications and breakdown supervision.

Devices

Different types of devices, used for field implementation of a given regulation method, can be classified according to the following partition rule: fixed device, manual device, motorised device, mechanical automatic device (e.g. AMIL, AVIS, AVIO gates), or electromechanical automatic device. Mechanical automatic gates are relatively cheap, sufficiently precise and of easy maintenance. They are very well adapted to small or medium size canals. Electromechanical gates are even more precise and flexible. Target values and control algorithms can be easily modified, and remote control can be introduced. However, they are more expensive, less reliable, and more demanding in maintenance. Usually, they are used on large canals.

1.6 Conclusions

Over the last decades, much research effort has been devoted to water flow control of open–channel conveyance systems such as irrigation channels. A great number of regulatory procedures have been proposed. In this chapter a very detailed classification is reported. It has been firstly proposed by Malaterre in [54], where he defined four essential characterization criteria:

- considered variables: controlled, measured and control action variables;
As it emerges immediately, it is nonsense to establish in general which is the most efficient regulation procedure. The best choice obviously depends on the physical and geometric characteristics of the open–channel at hand.

In this thesis we shall deal with the design of decentralized constant–volume controllers. In particular, we shall propose both proportional and proportional integral decentralized controllers. The requirement of decentralization origins from the large dimensions of the physical systems at hand, sometimes of the order of several kilometers. Moreover, the choice of volume variations as controlled variables origins from the requirement to reduce the transient time, in particular to reduce it to the time necessary to the settling of the new flow rate values in each canal reach. Finally, since volume variations are not measured variables, the design of an asymptotic state observer will be discussed. Obviously, even the observer has to be decentralized, i.e., each state variable has to be estimated on the base of only local measurements thus not vanishing the advantage of decentralization in the control.
Chapter 2

Dynamical models of open channels

2.1 Introduction

A wide variety of mathematical models describing the dynamic behaviour of open channels have been presented in the literature. Sometimes differences among them are really significant. In this chapter we first recall the main contributions on the subject. Thereafter, we discuss the fundamental steps in the deduction of the two approximate models used in this thesis for the synthesis of the proposed decentralized controllers. Both these models are the result of much research effort by Corriga et al. [18, 19, 21]. Then, a new linear decoupled dynamical model of open–channels, that will be useful when designing a decentralized volume variation observer, is derived. Finally, we briefly present the SIC (Simulation of Irrigation Canals) software used as a validating tool for numerical simulations.

2.2 State of art

In this section we provide a classification of the existing mathematical models describing the dynamic behaviour of open–channels.

- **Complete nonlinear models**: The Saint–Venant’s equations and the complementary internal and external boundary conditions can directly be used to design a controller. This is a very recent work [17] whose validity only holds in the case of limited systems. However, if the nonlinear effects are the main control difficulties, in some cases it can also be very powerful.

- **Complete linearized models**: A linearized version of these equations can also be used [11, 103]. This approach as the former nonlinear one, is powerful since it use the most complete version of the system model. But, the mathematical techniques required
to design the controller are most complex, and can be used, for the moment, only on homogeneous geometry.

- **Infinite order linear transfer functions**: Since the above described models are not very easy to use when designing a controller, some simplified models have been delivered from the previous ones through some simplifications. A linearization, Laplace transform and integration of the above Saint–Venant’s equations lead to a linear infinite order model \([7, 8]\), of the following form (in the frequency domain):

\[
\begin{pmatrix}
Q_x \\
Z_0
\end{pmatrix} =
\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
Q_0 \\
Z_x
\end{pmatrix},
\]

where \(Q_x\) is the discharge variation at distance \(x\), \(Z_x\) is the water depth at the same location, and \(Q_0\) and \(Z_0\) are the same values at the reference location \(x = 0\). The \(m_{ij}\) depend on the geometry and the hydraulic configuration of the canal system. This model has the advantage of keeping the distributed parameter system characteristics and therefore the infinite state space dimension. It supposes that the concerned system is homogeneous and at the uniform flow conditions.

Another infinite order linear transfer function model has been firstly presented by Corriga et al. in \([19, 21]\). The elements of the transfer function matrix are transcendental functions with respect to the Laplace variable \(s\). Even if in the literature it has been only presented as an intermediate step towards the deduction of a more simplificative linear state space model, in this thesis it has been successfully used for the design of a decentralized controller by means of an efficient numerical procedure which only implies the solution of a linear least squares problem. The deduction of this model, denoted in the following as **reference model**, is widely discussed in this chapter. It derives from the Saint-Venant equations linearized around a reference configuration of uniform flow.

- **Finite order nonlinear models**: A discretized version of Saint–Venant’s equations (in space, or time and space) can also be used \([51]\). A numerical approach is therefore used instead of a mathematical one. The main advantage is to simplify the control design and to allow this approach for almost any type of canal system, while keeping the nonlinear features of the system. The main limitation is that these models, based on numerical schemes such as the Preissmann scheme, required subcritical flow. The same approach can maybe be extended to supercritical flow, by using other schemes, but this was never tested according to the authors’ knowledge. The numerical scheme introduces a discrepancy in the modelization, but this type of model can still be considered as very precise.

- **Finite order linear models (state space models)**: The nonlinear or the infinite order feature of the previous models reduce the spectrum of control theories that can be used. In particular, all the methodologies developed in the field of LQ optimal controllers cannot be used. To allow this, a linear finite order state space model is required and can
be obtained (lumped model obtained from linearization of the Saint–Venant’s equations) [18, 19, 21].

- **Finite order linear models (transfer functions):** the advantage of the previous model is to be simple enough and cope with the multivariable feature of the considered systems. But, on large systems, it can be costly in terms of required data, memory space, computation time. In order to overcome this difficulty, some simpler linear MIMO or SISO transfer function models can be used. These models can be first order, second order or second order with delay depending on the size of the system and hydraulic conditions [54, 72, 81, 82].

- **Neural Network models:** Like in most industrial fields, neural network models and controllers were tested for irrigation canals or rivers [100, 101]. The identification phase is costly and difficult when the canal system is at the design stage. So far, these experiments have not been very successful.

- **Fuzzy models:** This type of model and corresponding controllers have also been designed for irrigation canals [10, 96, 97]. The advantage of this model is mainly to easily provide a nonlinear model of the system. But its quality is not as good as the previous ones. Also, for MIMO systems this approach seems difficult.

- **Petri net models:** The first application of this type of models to irrigation canals have been presented recently [22]. The advantage of this type of models is to allow the use of many techniques and tools that proved to be efficient in systems bearing a close resemblance to canal network systems. Nevertheless, no example of controller has been proposed yet.

As it has been already said before, in this thesis we shall deal with two dynamical models of open–channels that have been firstly deduced from Corriga et al. in [18, 19, 21]. Both of them have been derived from the Saint–Venant equations linearized around a reference configuration of uniform flow.

The first one, denoted in the following as the **reference model** is in terms of transcendental functions, thus it is an infinite order model. The output variables are the volume variations with respect to a reference configuration of uniform flow, while the input variables are the gate opening variations with respect to the same reference configuration.

On the contrary, the second one is a linear state space dynamical model and is obtained from the previous one by means of a Taylor series expansion around $s = 0$, where the most important phenomena occur, of an appropriate transfer function matrix. In the following it will be denoted as **approximate nominal model**. It is the smallest size state space model cited in the literature, being its order equal to the number of pools in the canal. The choice of such a model for the synthesis of the controller derives from the consideration that it seemed to be a good trade–off between accuracy and simplicity. State variables are equal to the volume
variations in the pools and control variables are gate opening variations. As it will be clarified in the following, all variations are computed with respect to a reference configuration of uniform flow. This model enables us to design a constant–volume control law. The choice of volume variations as controlled variables origins from the requirement to reduce the transient time, in particular to reduce it to the time necessary to the settling of the new flow rate values in each canal reach.

2.3 Dynamical models for the synthesis of the controllers

Consider the system sketched in figure 2.1, consisting of a channel of $n$ reaches joined by $n+1$ gates, where the last gate (the $(n+1)$-st) is fixed and the others are controlled. Let us suppose that water is conveyed to the first reach from a reservoir with constant level and that the level downstream from the final reach is also constant.

All the other variables considered, apart from those that define the geometry, represent the variations with respect to a reference configuration, assumed to be of uniform flow.

It has been assumed that the canal cross section is trapezoidal. Its scheme is shown in figure 2.2.

In the following sections we summarize the main steps in the deduction of the two mentioned models firstly deduced from Corriga et al. in [18, 19, 20, 21]. As already specified above, they have been derived from the linearization of the Saint–Venant equations around a reference configuration of uniform flow.

In particular, as it will be discussed in detail in the following, a state-space realization of
the form
\[ \dot{v}(t) = Av(t) + B\sigma(t) \]  \hspace{1cm} (2.1)
can be obtained, where \( v = (v_1 \cdots v_i \cdots v_n)^T \) is the state-vector and \( v_i \) represents the water volume variation in the \( i \)-th reach; \( \sigma = (\sigma_1 \cdots \sigma_i \cdots \sigma_n)^T \) is the input vector and each \( \sigma_i \) represents the variation in the \( i \)-th gate opening section.

By taking into account the users withdrawals, the above state equation can be rewritten as
\[ \dot{v}(t) = Av(t) + B\sigma(t) - Iq_C(t) \]  \hspace{1cm} (2.2)
where \( q_C = (q_{C1} \cdots q_{Ci} \cdots q_{CN})^T \) is the vector of unknown disturbances and \( q_{Ci} \) represents the variation of users withdrawals in the \( i \)-th reach; \( I \) is the \( n \)-order identity matrix.

### 2.3.1 Reference model

As it has been already underlined, the above linear model is derived from the Saint–Venant equations linearized around a reference condition of uniform flow \([104]\). Thus, water level depths and velocities are the same in each cross section of a reach.

The Saint-Venant equations are partial differential equations where the independent variables are the time \( t \) and the abscissa \( x \), and the spatial and time dependent variables are the water level depth variation \( h \) and the discharge variation \( q \) in each reach, with respect to the reference configuration of uniform flow. Thus, for the \( i \)-th reach, we can write
\[ h_i(x, t) = H_i(x, t) - H_i(0, t) \]  \hspace{1cm} (2.3)
\[ q_i(x, t) = Q_i(x, t) - Q_{0i} \]  \hspace{1cm} (2.4)
being independent from the abscissa \( x \) the discharge value in the uniform configuration. Note that capital letters have been used to denote absolute quantities, while small letters have been used to denote relative quantities.

Now, let us introduce the following notation:
• \(l_i\) length of the \(i\)-th reach;
• \(w_i\) canal bottom width in the \(i\)-th reach;
• \(p_1\) canal bottom slope;
• \(p_2\) canal side slope;
• \(H_{0i}\) water level depth in the \(i\)-th reach under uniform flow condition;
• \(W_i\) water surface width in the \(i\)-th reach;
• \(y_{0i}\) mean velocity of water in the \(i\)-th reach in reference condition;
• \(c_{0i}\) celerity of perturbation in reference condition.

Obviously, it holds that
\[
W_i = w_i + 2p_2 H_{0i}. \tag{2.5}
\]

Then, let us introduce the coefficient
\[
r_i = \frac{w_i}{H_{0i}} + \frac{p_2}{w_i + 2p_2} \tag{2.6}
\]
that is the ratio of water level depth in an equivalent rectangular section, where \(W_i\) is the water surface width, to actual level depth in the \(i\)-th reach. If we denote as \(g\) the gravitation acceleration, we have that
\[
c_{0i} = \sqrt{gr_i H_{0i}}. \tag{2.7}
\]

Thus, taking into account the above assumptions, the Saint-Venant equations can be written as
\[
\frac{\delta^2 h_i}{\delta t^2} + 2y_{0i} \frac{\delta^2 h_i}{\delta t \delta x} - (c_{0i}^2 - y_{0i}^2) \frac{\delta^2 h_i}{\delta x^2} + \beta_i \frac{\delta h_i}{\delta t} + \gamma_i \frac{\delta h_i}{\delta x} = 0 \tag{2.8}
\]
\[
\frac{\delta q_i}{\delta x} + W_i \frac{\delta h_i}{\delta t} = 0 \tag{2.9}
\]
where
\[
\beta_i = \frac{2g p_1}{y_{0i}} \tag{2.10}
\]
\[
\gamma_i = \frac{p_1 c_{0i}^2 r_i}{H_{0i}}. \tag{2.11}
\]

At time instant \(t = 0\), no variation with respect to the reference configuration exists, thus
\[
\begin{cases}
    h(x, 0) = 0 & \quad \dot{h}(x, 0) = 0 \\
    q(x, 0) = 0 & \quad \dot{q}(x, 0) = 0
\end{cases} \tag{2.12}
\]
and Laplace–transforming equations (2.8) and (2.9), we get the following differential equations
\[
(c_{0i}^2 - y_{0i}^2) \frac{\delta^2 H_i}{\delta x^2} + (2sy_{0i} + \gamma_i) \frac{\delta H_i}{\delta x} - s(s + \beta_i) H_i = 0 \tag{2.13}
\]
\[ \frac{\delta Q_i}{\delta x} + sW_i H_i = 0 \]  

(2.14)

where capital letters denote Laplace–transformed variables.

Solutions of the above equations are of the form

\[ H_i(x, s) = C_{1i} e^{\alpha_{1i} x} + C_{2i} e^{\alpha_{2i} x} \]  

(2.15)

\[ Q_i(x, s) = -sW_i \left( \frac{C_{1i}}{\alpha_{1i}} e^{\alpha_{1i} x} + \frac{C_{2i}}{\alpha_{2i}} e^{\alpha_{2i} x} \right) \]  

(2.16)

where \( \alpha_{1i} \) and \( \alpha_{2i} \) are the roots of the characteristic equation associated to (2.13)

\[ \alpha_{1i,2i} = \frac{2sy_0 + \gamma_i \pm \sqrt{(2sy_0 + \gamma_i)^2 + 4(c_{0i}^2 - y_{0i}^2)s(s + \beta_i)}}{2(c_{0i}^2 - y_{0i}^2)}. \]  

(2.17)

By imposing the boundary conditions

\[
\begin{aligned}
    & H_i(0, s) = H_{Ai}(s) \\
    & Q_i(0, s) = Q_{Ai}(s) \\
    & H_i(l_i, s) = H_{Bi}(s) \\
    & Q_i(l_i, s) = Q_{Bi}(s)
\end{aligned}
\]  

(2.18)

(2.19)

the following relationships can be obtained

\[ C_{1i} + C_{2i} = H_{Ai}(s) \]  

(2.20)

\[ -sW_i \left( \frac{C_{1i}}{\alpha_{1i}} + \frac{C_{2i}}{\alpha_{2i}} \right) = Q_{Ai}(s) \]  

(2.21)

\[ C_{1i} e^{\alpha_{1i} l_i} + C_{2i} e^{\alpha_{2i} l_i} + H_{Bi}(s) \]  

(2.22)

\[ -sW_i \left( \frac{C_{1i}}{\alpha_{1i}} e^{\alpha_{1i} l_i} + \frac{C_{2i}}{\alpha_{2i}} e^{\alpha_{2i} l_i} \right) = Q_{Bi}(s). \]  

(2.23)

The above equations can be used to express the water level depth variations at the upstream and the downstream of the \( i \)-th reach as a function of the corresponding discharge variations. In fact,

\[
\begin{aligned}
    H_{Ai}(s) &= \frac{H_{Ai}(s)}{Q_{Ai}(s)} \bigg|_{Q_{Bi} = 0} Q_{Ai} + \frac{H_{Ai}(s)}{Q_{Bi}(s)} \bigg|_{Q_{Ai} = 0} Q_{Bi} \\
    H_{Bi}(s) &= \frac{H_{Bi}(s)}{Q_{Ai}(s)} \bigg|_{Q_{Bi} = 0} Q_{Ai} + \frac{H_{Bi}(s)}{Q_{Bi}(s)} \bigg|_{Q_{Ai} = 0} Q_{Bi}
\end{aligned}
\]  

(2.24)

(2.25)

where the transfer functions have the following expressions

\[
\left. \frac{H_{Ai}(s)}{Q_{Ai}(s)} \right|_{Q_{Bi} = 0} = \frac{\alpha_{2i} e^{\alpha_{1i} l_i} - \alpha_{1i} e^{\alpha_{2i} l_i}}{sW_i (e^{\alpha_{2i} l_i} - e^{\alpha_{1i} l_i})}
\]  

(2.26)
\[ \frac{H_{Ai}(s)}{Q_{Bi}(s)} \bigg|_{Q_{Ai}=0} = \frac{\alpha_{2i} - \alpha_{1i}}{sW_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.27)
\[ \frac{H_{Bi}(s)}{Q_{Ai}(s)} \bigg|_{Q_{Bi}=0} = \frac{(\alpha_{2i} - \alpha_{1i})e^{(\alpha_{1i} + \alpha_{2i})l_i}}{sW_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.28)
\[ \frac{H_{Bi}(s)}{Q_{Bi}(s)} \bigg|_{Q_{Ai}=0} = \frac{\alpha_{1i}e^{\alpha_{1i}l_i} - \alpha_{2i}e^{\alpha_{2i}l_i}}{sW_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.29)

Now, if we assume
\[ A_{1i}(s) = \frac{\alpha_{2i}e^{\alpha_{1i}l_i} - \alpha_{1i}e^{\alpha_{2i}l_i}}{W_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.30)
\[ A_{2i}(s) = \frac{\alpha_{1i} - \alpha_{2i}}{W_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.31)
\[ A_{3i}(s) = \frac{(\alpha_{2i} - \alpha_{1i})e^{(\alpha_{1i} + \alpha_{2i})l_i}}{W_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.32)
\[ A_{4i}(s) = \frac{\alpha_{1i}e^{\alpha_{1i}l_i} - \alpha_{2i}e^{\alpha_{2i}l_i}}{W_i(e^{\alpha_{2i}l_i} - e^{\alpha_{1i}l_i})} \] (2.33)
we can write
\[
\begin{align*}
H_{Ai}(s) &= \frac{A_{1i}(s)Q_{Ai}(s)}{s} + \frac{A_{2i}(s)Q_{Bi}(s)}{s} \\
H_{Bi}(s) &= \frac{A_{3i}(s)Q_{Ai}(s)}{s} + \frac{A_{4i}(s)Q_{Bi}(s)}{s}
\end{align*}
\] (2.34)

A further relation which is necessary to completely define the mathematical model of an open channel, is the physical law governing the flow of water under the gates [104]. In the case of head lights (where the outline of the outflow section is completely submerged), during the regime of permanent flow, the discharge is related to the total surface of the outflow section and to the difference between the upstream and downstream level of the light itself.

If we denote as \( \Sigma \) the opening section of a gate, we can write
\[ Q = \eta \Sigma \sqrt{2g(H_B - H_A)} \] (2.35)

where \( \eta \) is the discharge coefficient. \( H_A \) and \( H_B \) are the water level depths at the upstream and downstream of the gate, thus they are absolute quantities like the discharge \( Q \) and the surface \( \Sigma \).

If only small variations of the gate opening section with respect to the uniform condition occur, we can evaluate the derivative of \( Q \) as follows
\[
dQ = \delta Q \left|_{\Sigma} \right|_{P_0} d\Sigma + \delta Q \left|_{H_B} \right|_{P_0} dH_B + \delta Q \left|_{H_A} \right|_{P_0} dH_A \] (2.36)
where $P_0 = [\Sigma_0, H_{B0}, H_{A0}]$ is the initial point, relative to the uniform condition, where derivatives are evaluated. If we define

$$a = \frac{\delta Q}{\delta \Sigma}|_{P_0}, \quad b = \frac{\delta Q}{\delta H_B}|_{P_0}, \quad c = \frac{\delta Q}{\delta H_A}|_{P_0},$$

(2.37)
equation (2.36) can be rewritten as

$$dQ = a \, d\Sigma + b \, dH_B + c \, dH_A.$$  

(2.38)

It is easy to prove that

$$a = \eta \sqrt{2g(H_{B0} - H_{A0})}$$

(2.39)

$$b = \frac{\eta \Sigma_0 \sqrt{2g}}{2\sqrt{H_{B0} - H_{A0}}}$$

(2.40)

$$c = -b.$$  

(2.41)

Equation (2.38) can be rewritten for the $i$-th reach and taking into account that small letters are used to denote variations with respect to the reference condition of uniform flow. Therefore,

$$q_{Ai}(t) = a_i \sigma_i(t) + b_i h_{B_{i-1}}(t) + c_i h_{A_i}(t).$$

(2.42)

This equation shows that, under appropriate assumptions, upstream discharge variation in the $i$-th reach is linearly related to the gate opening variation and to the upstream and downstream water level depths in the $i$-th reach. The above coefficients are defined as

$$a_i = \eta \sqrt{2g(H_{0i-1} - H_{0i})}$$

(2.43)

$$b_i = \frac{\eta \Sigma_{0i} \sqrt{2g}}{2\sqrt{H_{0i-1} - H_{0i}}}$$

(2.44)

$$c_i = -b_i$$

(2.45)

where the subscript 0 denotes the reference configuration.

Equation (2.42) can be rewritten in the Laplace domain as

$$Q_{Ai}(s) = a_i \Sigma_i(s) + b_i H_{Bi-1}(s) + c_i H_{Ai}(s).$$

(2.46)

Note that if no variation on users withdrawals occur, then

$$Q_{Ai}(s) = Q_{Bi-1}(s), \quad i = 2, 3, \ldots, n + 1.$$  

(2.47)
Now, let us define the following vectors

\[
H = \begin{bmatrix}
H_{A1} \\
H_{B1} \\
\vdots \\
H_{Ai} \\
H_{Bi} \\
\vdots \\
H_{An} \\
H_{Bn}
\end{bmatrix}, \quad
Q = \begin{bmatrix}
Q_{A1} \\
\vdots \\
Q_{Ai} \\
\vdots \\
Q_{A_{n+1}}
\end{bmatrix}, \quad
\Sigma = \begin{bmatrix}
\Sigma_1 \\
\vdots \\
\Sigma_i \\
\vdots \\
\Sigma_n
\end{bmatrix}, \quad
V = \begin{bmatrix}
V_1 \\
\vdots \\
V_i \\
\vdots \\
V_n
\end{bmatrix}
\]

(2.48)

where

- \(H\) is a 2\(n\) vector whose components represent the Laplace-transform of the upstream and downstream water level variations in the \(n\) reaches;
- \(Q\) is an \(n + 1\) vector whose components represent the Laplace-transform of the discharge variations in the \(n + 1\) gates;
- \(\Sigma\) is an \(n\) vector whose components represent the Laplace-transform of the \(n\) gate opening sections;
- \(V\) is an \(n\) vector whose components represent the Laplace-transform of the water volume variations in the \(n\) reaches.

Note that all variations are evaluated with respect to the reference configuration of uniform flow.

Now, scalar equations (2.34) and (2.46) can be rewritten in vectorial form

\[
H(s) = \frac{1}{s} \tilde{A}(s) Q(s)
\]

(2.49)

\[
Q(s) = \Gamma \Sigma(s) + \Delta H(s)
\]

(2.50)

where matrices \(\tilde{A}, \Gamma\) and \(\Delta\) haven’t been defined yet.

\(\tilde{A}(s)\) is a 2\(n\) \(\times\) (\(n + 1\)) matrix with the following structure

\[
\tilde{A}(s) = \begin{bmatrix}
A_{11}(s) & A_{21}(s) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
A_{31}(s) & A_{41}(s) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & A_{12}(s) & A_{22}(s) & 0 & \cdots & \cdots & \cdots & 0 \\
0 & A_{32}(s) & A_{42}(s) & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & A_{4(n-1)}(s) & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & A_{1n}(s) & A_{2n}(s) & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & A_{3n}(s) & A_{4n}(s) & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

(2.51)
and each reach is characterized by the matrix block

\[
\begin{bmatrix}
A_{11}(s) & A_{2i}(s) \\
A_{3i}(s) & A_{4i}(s)
\end{bmatrix}, \tag{2.52}
\]

\(\Gamma\) is a \((n+1) \times n\) constant matrix defined as

\[
\Gamma = \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & a_N \\
0 & \cdots & 0
\end{bmatrix}, \tag{2.53}
\]

while \(\Delta\) is a \((n+1) \times 2n\) constant matrix with the following structure

\[
\Delta = \begin{bmatrix}
c_1 & 0 & \cdots & 0 \\
0 & b_2 & c_2 & 0 & \cdots & 0 \\
0 & 0 & b_3 & c_3 & 0 & \cdots & 0 \\
\vdots & & & & & \\
0 & \cdots & 0 & b_{N+1}
\end{bmatrix}. \tag{2.54}
\]

To obtain an expression of equation (2.49) that still holds for \(s = 0\), equation (2.49) can be rewritten as

\[
H(s) = A_1(s)V(s) + A_2(s)Q(s) \tag{2.55}
\]

where the \(2n \times n\) matrix \(A_1(s)\) and the \(2n \times (n+1)\) matrix \(A_2(s)\) are defined below

\[
A_1(s) = \begin{bmatrix}
A_{11}(s) & 0 & \cdots & 0 \\
-A_{41}(s) & 0 & \cdots & 0 \\
0 & A_{12}(s) & 0 & \cdots & 0 \\
0 & -A_{42}(s) & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & A_{1n}(s) \\
0 & \cdots & 0 & -A_{4n}(s)
\end{bmatrix}, \tag{2.56}
\]
A_2(s) = \begin{bmatrix}
0 & \frac{A_{11}(s) + A_{21}(s)}{s} & \cdots & \cdots & 0 \\
\frac{A_{31}(s) + A_{41}(s)}{s} & 0 & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & A_{32}(s) + A_{42}(s) & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \frac{A_{3n}(s) + A_{4n}(s)}{s} \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}.

(2.57)

To characterize the properties of matrices A_1(s) and A_2(s) for s → 0, the limit behaviour of the above quantities should be evaluated.

The roots of the characteristic equation satisfies the following asymptotic conditions

\[ \alpha_{1i} = \lim_{s \to 0} \alpha_{1i}(s) = \frac{\gamma_i}{c_{0i}^2 - y_{0i}^2} \]  

(2.58)

\[ \alpha_{2i} = \lim_{s \to 0} \alpha_{2i}(s) = 0. \]  

(2.59)

Therefore, the constant gains K_{ji} can be determined

\[ K_{1i} = \lim_{s \to 0} A_{1i}(s) = -\frac{\alpha_{1i}}{W_i(1 - e^{\alpha_{1i}t_i})} \]  

(2.60)

\[ K_{2i} = \lim_{s \to 0} A_{2i}(s) = \frac{\alpha_{1i}}{W_i(1 - e^{\alpha_{1i}t_i})} = -K_{1i} \]  

(2.61)

\[ K_{3i} = \lim_{s \to 0} A_{3i}(s) = -\frac{\alpha_{1i}e^{\alpha_{1i}t_i}}{W_i(1 - e^{\alpha_{1i}t_i})} \]  

(2.62)

\[ K_{4i} = \lim_{s \to 0} A_{4i}(s) = \frac{\alpha_{1i}e^{\alpha_{1i}t_i}}{W_i(1 - e^{\alpha_{1i}t_i})} \]  

(2.63)

Thus, it holds that

\[ A_1(0) = \begin{bmatrix}
K_{11} & 0 & \cdots & 0 \\
K_{31} & 0 & \cdots & 0 \\
0 & K_{12} & 0 & \cdots & 0 \\
0 & K_{32} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & K_{1n} \\
0 & \cdots & 0 & K_{3n} \\
\end{bmatrix}. \]  

(2.64)

When dealing with matrix A_2(s), we get the following expressions

\[ \lim_{s \to 0} \frac{A_{1i}(s) + A_{2i}(s)}{s} = \frac{\beta_i}{\gamma_i} \left( \frac{1}{W_i} - K_{1i} \right) \]  

(2.65)
\[
\lim_{s \to 0} \frac{A_{3i}(s) + A_{4i}(s)}{s} \beta_i \left( \frac{1}{\gamma_i} - K_{3i}t_i \right) = 0
\] (2.66)

that enables us to sustain that also matrix \( A_2(s) \) is defined for \( s \to 0 \).

Summarizing, we can affirm that the equations necessary for the deduction of the reference model in the Laplace-domain, are

\[
H(s) = A_1(s)V(s) + A_2(s)Q(s) \\
Q(s) = \Gamma \Sigma(s) + \Delta H(s).
\]

Substituting the first equation in the second equation, we obtain

\[
Q(s) = [I - \Delta A_2(s)]^{-1} \Delta A_1(s) V(s) + [I - \Delta A_2(s)]^{-1} \Gamma \Sigma(s) \tag{2.67}
\]

and setting

\[
A_3(s) = [I - \Delta A_2(s)]^{-1} \Delta A_1(s) \tag{2.68}
\]

and

\[
B_3(s) = [I - \Delta A_2(s)]^{-1} \Gamma \tag{2.69}
\]

equation (2.67) can be rewritten as

\[
Q(s) = A_3(s)V(s) + B_3(s)\Sigma(s). \tag{2.70}
\]

Left-multiplying both sides of equation (2.70) by the following \( n \times (n+1) \)-dimensional operator

\[
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 \\
\vdots \\
0 & \cdots & 0 & 1 & -1
\end{bmatrix}
\]

gives

\[
\Gamma^*Q(s) = sV(s) = \Gamma^*A_3(s)V(s) + \Gamma^*B_3(s)\Sigma(s) \tag{2.71}
\]

and setting

\[
A(s) = \Gamma^*A_3, \quad B(s) = \Gamma^*B_3 \tag{2.72}
\]

we get

\[
sV(s) = A(s)V(s) + B(s)\Sigma(s) \tag{2.73}
\]

from which the transfer matrix

\[
G(s) = [sI - A(s)]^{-1}B(s) \tag{2.74}
\]

is obtained. Its entries are transcendental functions of \( s \).
2.3.2 Approximate nominal model

In this section we show how it is possible to derive a linear time–invariant model to describe the open–channel dynamics. Obviously, this model is less realistic than the previous one. However its importance is related to the fact that it allows us to use the classical control techniques to derive suitable control laws whose validity will be proved implementing them on a completely non–linear model of the system under examination.

At this purpose, let us consider once again the equation (2.49). It can be expanded in Taylor series, being the elements of $\tilde{A}(s)$ analytic functions. Since the model needs to hold mainly in the low-frequency range, where the most significant phenomena take place, $s = j\omega = 0$ is taken as initial point. By truncating the series expansion to the second term, equation (2.49) becomes

$$
H(s) = \frac{1}{s} \left[ \tilde{A}(0) + s \frac{d\tilde{A}(s)}{ds} \right] Q(s) = \frac{1}{s} \tilde{A}(0)Q(s) + \tilde{A}'(0)Q(s)
$$

(2.75)

where

$$
\tilde{A}'(0) = \frac{d\tilde{A}(s)}{ds} \bigg|_{s=0}
$$

(2.76)

At this point, it is easy to verify that the following equalities hold

$$
A_{11}(0) = -A_{21}(0) \tag{2.77}
$$

$$
A_{31}(0) = -A_{41}(0) \tag{2.78}
$$

thus, we can write

$$
\frac{1}{s} \tilde{A}(0)Q(s) = \frac{1}{s} \begin{bmatrix}
A_{11}(0) & A_{21}(0) & 0 & 0 & \cdots & \cdots & 0 \\
A_{31}(0) & A_{41}(0) & 0 & 0 & \cdots & \cdots & 0 \\
0 & A_{12}(0) & A_{22}(0) & 0 & \cdots & \cdots & 0 \\
0 & A_{32}(0) & A_{42}(0) & 0 & \cdots & \cdots & 0 \\
\vdots & & & & & & \\
0 & \cdots & \cdots & A_{4(n-1)}(0) & 0 & \\
0 & \cdots & \cdots & A_{1n}(0) & A_{2n}(0) \\
0 & \cdots & \cdots & A_{3n}(0) & A_{4n}(0)
\end{bmatrix}
\begin{bmatrix}
Q_{A1}(s) \\
Q_{A2}(s) \\
\vdots \\
Q_{A_{n+1}}(s)
\end{bmatrix}
$$

(2.79)

Carefully examining the above structure, it can be noted that each non–null element is obtained by multiplying the matrix block

$$
\begin{bmatrix}
A_{11}(0) & -A_{11}(0) \\
A_{31}(0) & -A_{31}(0)
\end{bmatrix}
$$

(2.80)

by the sub–vector

$$
\begin{bmatrix}
Q_{A1}(s) \\
Q_{A_{i+1}}(s)
\end{bmatrix}
$$

(2.81)
thus getting the following scalar relationships
\[
\frac{1}{s} A_{1i}(0) [Q_{Ai}(s) - Q_{Ai+1}(s)] = A_{1i}(0) V_i(s) = K_{1i} V_i(s) 
\] (2.82)
\[
\frac{1}{s} A_{3i}(0) [Q_{Ai}(s) - Q_{Ai+1}(s)] = A_{3i}(0) V_i(s) = K_{3i} V_i(s) 
\] (2.83)
that can be rewritten in a more compact form as
\[
\frac{1}{s} \tilde{A}(0) Q(s) = A(0) V(s) 
\] (2.84)
where
\[
A(0) = \begin{bmatrix}
K_{11} & 0 & \cdots & 0 \\
K_{31} & 0 & \cdots & 0 \\
0 & K_{12} & 0 & \cdots & 0 \\
0 & K_{32} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & K_{1n} \\
0 & \cdots & 0 & K_{3n}
\end{bmatrix} 
\] (2.85)

Now, equation (2.75) can be rewritten as
\[
H(s) = A(0) V(s) + \tilde{A}'(0) Q(s) 
\] (2.86)
where the elements of matrix \( \tilde{A}'(0) \) haven’t been defined yet. At this purpose, let us consider again the roots of the characteristic equation defined in (2.17) and let
\[
\alpha_{1io}' = \left. \frac{d\alpha_{1io}(s)}{ds} \right|_{s=0} = \frac{2 y_0 i}{(c_0^2 - y_0^2)} + \frac{\beta_i}{\gamma_i} 
\] (2.87)
\[
\alpha_{2io}' = \left. \frac{d\alpha_{2io}(s)}{ds} \right|_{s=0} = -\frac{\beta_i}{\gamma_i}. 
\] (2.88)
The derivative of the expressions (2.30)–(2.31)–(2.32)–(2.33), evaluated for \( s = 0 \), are the following
\[
A_{1i}'(0) = \left. \frac{dA_{1i}(s)}{ds} \right|_{s=0} = \frac{(\alpha_{2io}' e^{\alpha_{1io}' l_i} - \alpha_{1io}' - \alpha_{1io} \alpha_{2io}' l_i)(1 - e^{\alpha_{1io}' l_i})}{W_i(1 - e^{\alpha_{1io}' l_i})^2} + \frac{\alpha_{1io} \alpha_{2io}' l_i(\alpha_{2io}' - \alpha_{1io}' e^{\alpha_{1io}' l_i})}{W_i(1 - e^{\alpha_{1io}' l_i})^2} 
\] (2.89)
\[
A_{2i}'(0) = \left. \frac{dA_{2i}(s)}{ds} \right|_{s=0} = \frac{(\alpha_{1io}' - \alpha_{1io}' - \alpha_{2io})(1 - e^{\alpha_{1io}' l_i}) - \alpha_{1io} \alpha_{2io}' l_i(\alpha_{2io}' - \alpha_{1io}' e^{\alpha_{1io}' l_i})}{W_i(1 - e^{\alpha_{1io}' l_i})^2} 
\] (2.90)
\[
A_{3i}'(0) = \left. \frac{dA_{3i}(s)}{ds} \right|_{s=0} = -A_{2i}'(0) e^{\alpha_{1io}' l_i} - A_{2i}(0) (\alpha_{1io}' + \alpha_{2io}' l_i) e^{\alpha_{1io}' l_i} 
\] (2.91)
\[ A_i'(0) = \left. \frac{dA_i(s)}{ds} \right|_{s=0} = \frac{(\alpha'_{1i} e^{\alpha_{1i}l_{i0}} + \alpha_{1i} \alpha'_{1i} l e^{\alpha_{1i}l_{i0}} - \alpha_{2i} e^{\alpha_{1i}l_{i0}})}{W_i(1 - e^{\alpha_{1i}l_{i0}})^2} - \frac{\alpha_{1i} l e^{\alpha_{1i}l_{i0}} (\alpha_{2i} e^{\alpha_{1i}l_{i0}} - \alpha_{1i} l e^{\alpha_{1i}l_{i0}})}{W_i(1 - e^{\alpha_{1i}l_{i0}})^2} \]

(2.92)

and define the \( i \)-th matrix block

\[
\begin{bmatrix}
A_{1i}'(0) & A_{2i}'(0) \\
A_{3i}'(0) & A_{4i}'(0)
\end{bmatrix}
\]

(2.93)

characteristic of matrix \( \tilde{A}'(0) \).

Following the same procedure as for the reference model, the expressions for the constant matrices \( A \) and \( B \) of equation (2.1) are obtained

\[ A = I^* [I - \Delta \tilde{A}'(0)]^{-1} \Delta A(0) \]

(2.94)

\[ B = I^* [I - \Delta \tilde{A}'(0)]^{-1} \Gamma \]

(2.95)

and the corresponding state equation in the Laplace domain is

\[ sV(s) = AV(s) + B \Sigma(s). \]

(2.96)

The associated transfer function matrix is

\[ G_A(s) = [sI - A]^{-1} B \]

(2.97)

whose elements are rational functions of \( s \), which is an approximation of (2.74).

Inverse–transforming equation (2.96), the state variable equation (2.1) is obtained.

It is important to note that the above model is valid only under the assumption of null withdrawals, i.e., only if

\[ Q_{Ai}(s) = Q_{Bi-i}(s) \quad i = 2, 3, \ldots, n+1. \]

(2.98)

If disturbances are taken into account, the above equation is not yet valid. In fact,

\[ Q_{Ai}(s) = Q_{Bi-1}(s) - Q_{Ci-j}(s) \quad i = 2, 3, \ldots, n+1, \]

(2.99)

i.e., the input flow in the \( i \)-th reach is equal to the difference between the output flow in the \((i - 1)\)-th reach and the flow required from users. In fact, it has been assumed that users withdrawals are situated at the end of each reach.

Now, let us define the following \( n \) dimensional vector

\[ Q_C = \begin{bmatrix} Q_{C1} \\
\vdots \\
Q_{Ci} \\
\vdots \\
Q_{Cn} \end{bmatrix} \]

(2.100)
whose entries are the variation of users withdrawals with respect to the reference condition of uniform flow. Now, the state variable approximate model can be rewritten as

\[ sV(s) = AV(s) + B(s)\Sigma(s) - IQ_C(s) \]  \hspace{1cm} (2.101)

whose inverse-transforming provides the desired state equation (2.2).

### 2.4 A linear decoupled model for the synthesis of a decentralized observer

In this section we formulate a new linear decoupled dynamical model of open–channels. The main advantage of this model is that it enables the design of a decentralized volume variation observer. In fact, open–channels, like all large scale systems, require decentralization in the control. However, since in real applications state variables, i.e., volume variations in the pools, are not directly measurable, the design of a state estimator become an essential requirement. Obviously, such an observer should provide an estimation of the volume variation in each canal reach on the base of only local measurements, thus not vanishing the advantage of decentralization in the control.

At this purpose, let us consider again equation (2.50)

\[ Q(s) = \Gamma\Sigma(s) + \Delta H(s) \]

whose \( i \)-th row can be rewritten as

\[ Q_{Ai}(s) = a_i\Sigma_i(s) + b_iH_{Bi-1} + c_iH_{Ai}(s). \]  \hspace{1cm} (2.102)

Inverse L-transforming equation (2.102), we get

\[ q_{Ai}(t) = a_i\sigma_i(t) + b_ih_{Bi-1} + c_ih_{Ai}(t). \]  \hspace{1cm} (2.103)

Then, if we assume that no variation on users withdrawals occur, we can also write

\[ q_{Bi}(t) = a_{i+1}\sigma_{i+1}(t) + b_{i+1}h_{Bi} + c_{i+1}h_{Ai+1}(t). \]  \hspace{1cm} (2.104)

Now, let us consider equation (2.86)

\[ H(s) = A_1(0)V(s) + \tilde{A}'(0)Q(s). \]

By taking into account the structure of matrices \( A_1(0) \) and \( \tilde{A}'(0) \) we can write

\[
\begin{bmatrix}
H_{Ai}(s) \\
H_{Bi}(s)
\end{bmatrix} = 
\begin{bmatrix}
K_{1i} \\
K_{3i}
\end{bmatrix} V_i(s) + 
\begin{bmatrix}
A'_{1i}(0) & A'_{2i}(0) \\
A'_{3i}(0) & A'_{4i}(0)
\end{bmatrix}
\begin{bmatrix}
Q_{Ai}(s) \\
Q_{Bi}(s)
\end{bmatrix}, \hspace{1cm} i = 1, \cdots, n, \hspace{1cm} (2.105)
\]
or equivalently, in the time domain,
\[
\begin{bmatrix}
  h_{A_i}(t) \\
  h_{B_i}(t)
\end{bmatrix} = \begin{bmatrix}
  K_{i1} \\
  K_{3i}
\end{bmatrix} v_i(t) + \begin{bmatrix}
  A'_{1i}(0) & A'_{2i}(0) \\
  A'_{3i}(0) & A'_{4i}(0)
\end{bmatrix} \begin{bmatrix}
  q_{A_i}(t) \\
  q_{B_i}(t)
\end{bmatrix}, \quad i = 1, \ldots, n. 
\tag{2.106}
\]

Substituting equations (2.103) and (2.104) into (2.106), we obtain the following matrix equation
\[
\begin{bmatrix}
  h_{A_i}(t) \\
  h_{B_i}(t)
\end{bmatrix} = \overline{C}_i v_i(t) + \overline{D}_i \begin{bmatrix}
  \sigma_i(t) \\
  \sigma_{i+1}(t) \\
  h_{A_i+1}(t) \\
  h_{B_i-1}(t)
\end{bmatrix}, \quad i = 1, \ldots, n. 
\tag{2.107}
\]

where
\[
\overline{C}_i = \begin{bmatrix}
  1 - A'_{1i}(0)c_i & -A'_{2i}(0)b_{i+1} \\
  -A'_{3i}(0)c_i & 1 - A'_{4i}(0)b_{i+1}
\end{bmatrix}^{-1} \begin{bmatrix}
  K_{i1} \\
  K_{3i}
\end{bmatrix},
\tag{2.108}
\]
\[
\overline{D}_i = \begin{bmatrix}
  1 - A'_{1i}(0)c_i & -A'_{2i}(0)b_{i+1} \\
  -A'_{3i}(0)c_i & 1 - A'_{4i}(0)b_{i+1}
\end{bmatrix}^{-1} \begin{bmatrix}
  A'_{1i}(0)a_i & A'_{2i}(0)a_{i+1} & A'_{3i}(0)a_i & A'_{4i}(0)c_{i+1} & A'_{4i}(0)b_{i+1} \\
  A'_{2i}(0)a_i & A'_{3i}(0)a_i & A'_{4i}(0)c_{i+1} & A'_{4i}(0)b_{i+1}
\end{bmatrix}.
\tag{2.109}
\]

Then, under the assumption of no external disturbance, it holds that
\[
\dot{v}_i(t) = q_{A_i}(t) - q_{B_i}(t),
\]
where \(q_{A_i}\) and \(q_{B_i}\) are defined in (2.103) and (2.104), respectively. Therefore, by taking into account the above equation (2.107), we get
\[
\dot{v}_i = \overline{A}_i v_i + \overline{B}_i \begin{bmatrix}
  \sigma_i \\
  \sigma_{i+1} \\
  h_{A_i+1} \\
  h_{B_i-1}
\end{bmatrix}, \quad i = 1, \ldots, n. 
\tag{2.110}
\]

where
\[
\overline{A}_i = \begin{bmatrix}
  c_i \\
  -b_{i+1}
\end{bmatrix} \cdot \overline{C}_i,
\tag{2.111}
\]
\[
\overline{B}_i = \begin{bmatrix}
  c_i \\
  -b_{i+1}
\end{bmatrix} \cdot \overline{D}_i + \begin{bmatrix}
  a_i \\
  -a_{i+1} \quad -c_{i+1} \quad b_i
\end{bmatrix}. \tag{2.112}
\]

Now, let us define
\[
\overline{u}_i(t) = \begin{bmatrix}
  \sigma_i(t) \\
  \sigma_{i+1}(t) \\
  h_{A_i+1}(t) \\
  h_{B_i-1}(t)
\end{bmatrix}, \quad \overline{y}_i(t) = \begin{bmatrix}
  h_{A_i}(t) \\
  h_{B_i}(t)
\end{bmatrix}.
\]

In this way we get the \(n\) following linear independent systems
\[
\begin{align*}
\dot{v}_i(t) &= \overline{A}_i v_i(t) + \overline{B}_i(t)\overline{u}_i(t) \\
\overline{y}_i(t) &= \overline{C}_i v_i(t) + \overline{D}_i(t)\overline{u}_i(t)
\end{align*} \quad i = 1, \ldots, n. \tag{2.113}
\]
The state variable of the $i$-th system is equal to the volume variation in the $i$-th reach; the inputs are the opening variations of the gates delimiting the pool ($\sigma_i, \sigma_{i+1}$), the upstream water level variation of the upstream gate ($h_{Ai+1}$) and the downstream water level variation of the downstream gate ($h_{Bi-1}$); finally, the output variables are equal to the upstream ($h_{Ai}$) and downstream ($h_{Bi}$) water level variation in the $i$-th reach.

Note that the $n$ linear independent systems (2.113) can be rewritten in the more compact form, as

\begin{align}
\dot{v} &= \bar{A}v + \bar{B}\bar{u} \\
\bar{y} &= \bar{C}v + \bar{D}\bar{u}
\end{align}

(2.114)

where

\begin{align*}
\bar{A} &= \text{block diag} \bar{A}_i, \\
\bar{B} &= \text{block diag} \bar{B}_i, \\
\bar{C} &= \text{block diag} \bar{C}_i, \\
\bar{D} &= \text{block diag} \bar{D}_i,
\end{align*}

and

\[
v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_n \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix}.
\]

### 2.5 SIC, a simulation model for canal automation design

The SIC (Simulation of Irrigation Canals) [56] is one of the latest hydraulic models developed by Cemagref (Montpellier, France).

The first developments on hydraulic numerical modeling started at Cemagref in the early 1970’s. Lots of improved and updated versions have been made since this period. Right now, several hydraulic models exist at Cemagref, depending on the type and events to simulate (rivers, irrigation canals, dam break, drainage systems, etc.). One of these models has been particularly dedicated to irrigation canals. This model, called SIC, has been adapted from another hydraulic model, where some features have been removed, new ones have been introduced, and for which special user-friendly interfaces have been developed. It was also adapted to run on currently available IBM PC computers or compatibles.

The SIC hydraulic model is solving the complete Saint Venant equations. It uses the classical implicit Preissmann scheme. The implicit coefficient $\theta$ is set to 0.6. The time step can be selected from 0.01 to 999.99 minutes. The distance step can be chosen by the user.

Different studies have been carried out in order to evaluate the computational accuracy of the SIC model (or of previous versions). It is impossible to synthesize this accuracy issue in a few general words, since it depends on the type of modeled system, on its size, on the simulated hydraulic conditions, on the number of data entered, on their precision, etc.
In this context the main importance of SIC lies on the fact that automatic control of any structure (cross structure, offtakes, boundary conditions) is possible inside SIC. In addition, an interface with the commercial package MatLab allows to write directly the regulation modules as a MatLab function.

As it will be clear in the following, in absence of validation in the real field, all numerical simulations will be carried out on this software.

2.6 Conclusions

In this chapter we dealt with the problem of open–channels modeling. As it has been outlined in Section 2.2, a wide variety of mathematical models describing the dynamic behaviour of open channels have been proposed in the literature and significant differences among them exist. A state of art of the problem is presented and advantages and disadvantages of each model are discussed.

Then, we recall the fundamental steps in the deduction of two dynamical models used in this thesis for the synthesis of the controller. These models, denoted as reference model and approximate nominal model respectively, have been firstly deduced from the Saint–Venant equations by Corriga et al. in [18, 19, 21] and have been taken into account here, as they seem to be a good trade–off between accuracy and simplicity. The first model expresses the dynamic relationships, in terms of transcendental functions, between the gate opening sections and the corresponding stored water volume variations in the different canal reaches with respect to a reference configuration of uniform flow. Then, series expansion around $s = 0$ gives a state variable linear and time invariant model as well as the corresponding rational transfer matrix.

Thereafter, we show how a new linear decoupled dynamical model of open–channels can be formulated. The main advantage of this model is that it enables the design of a decentralized volume variation observer that provides an estimation of the volume variation in each canal reach on the base of only local measurements, thus not vanishing the advantage of decentralization in the control.

In the last section of this chapter a brief discussion on the commercial SIC software, that has been used as a validating tool for numerical simulations, is also presented.
Chapter 3

Decentralized control laws via parameter optimization

3.1 Introduction

It frequently happens that the structure of a controller is constrained in some way. In the process industries, for example, there are considerable costs associated with each non–zero element of the controller’s transfer–function matrix. Every such elements represents a considerable amount of hardware, possibly extensive, additional wiring, additional testing and maintenance procedures, and additional training for the plant’s operating personnel. There are therefore strong pressures to keep the number of ‘cross-couplings’ as low as possible.

Such constraints are not respected by classical design techniques for multivariable systems, except for the direct and inverse Nyquist-array techniques when driven entirely manually (so that the designer has explicit control over the controller’s structure) [53], and the QFT approach [40, 41, 42]. The designer using the approximate commutative controller approach can control the McMillan degree of the controller, but not the distribution of the dynamics over the controller’s elements: in general, each element of the controller’s transfer function will have the same complexity. With the LQG and $H_{\infty}$ approaches he loses control of the McMillan degree as well, and cannot enforce the open–loop stability of the controller. He must therefore either split the design problem into sub–problems which do not have structure constraints, apply these design techniques to the sub–problems, or abandon the theoretically based techniques.

\footnote{The QFT approach assumes that the plant uncertainty is represented by a set of templates [42] on the complex plane, each of which encloses within it all the possible frequency responses $g_{ij}(j\omega_k)$ from some input $j$ to some output $i$ at some frequency $\omega_k$. It also assumes that the design specification is in the form of bounds on the magnitude of the elements of the frequency–response matrices $T(j\omega)$ (the closed–loop transfer matrix) and/or $S(j\omega)$ (the sensitivity function). The QFT technique leads to a design which satisfies these specifications for all permissible plant variations, while approximately minimizing the transmission of output sensor noise (if such design is possible).}
and adopt an \textit{ad hoc} approach.

In this chapter we discuss in detail some numerical procedures for the derivation of a structured control law.

Firstly, we propose a really simple procedure for the design of a decentralized output feedback control law \cite{88} which, even if it presents some drawbacks with respect to much more sophisticated ones \cite{29, 30, 52}, has the considerable advantage that its computational complexity weakly increases with the dimension of the problem. It is inspired by previous research works carried out by the author in the field of the identification of unknown parameters in particular classes of systems of ordinary differential equations \cite{73, 87, 91}. It consists in the evaluation of the decentralized feedback gains such that the closed–loop behaviour of the decentralized system, approaches, in the least squares sense, the state evolution of a target centralized closed–loop plant (obviously, the target behaviour can also be chosen in a completely different way). Then, the synthesis process reduces to the problem of identifying a set of unknown parameters in a linear system of ordinary differential equations. Since also the time derivatives of the target states can be computed, such a problem can be solved as a simple linear least squares problem, where no difficulty due to local minima and good initial estimate occurs. A criterion for the choice of the observational data for the identification process is also provided.

The second design technique has been firstly proposed by Edmund in \cite{31} and clearly summarized in \cite{53}. It involves the solution of a linear least squares problem in the frequency–domain and this makes it applicable to a wide variety of problems, even characterized by non-rational transfer matrices. In particular, in the case at hand \cite{86, 90}, it is the only synthesis procedure that enables us to use the reference model of open–channels, whose derivation has been discussed in detail in Chapter 2.

Finally, we propose an original methodology for the design of a structured control law based on the minimization of the $H_2$ norm of a suitable transfer function matrix \cite{84, 85, 89}. This procedure is the equivalent in the frequency domain, of the LQR technique in the time domain. However, the advantage of such an approach is the possibility of establishing beforehand the gain matrix structure. However, from a numerical point of view, it is surely less efficient than the previous ones, since it requires the solution of a non–convex optimization problem and difficulties due to local minima may occur. Moreover, the only linear time–invariant model can be used for the synthesis of the structured controller.

Note that all these procedures can be easily extended to discrete–time systems.

In all cases examined results of numerical simulations are also presented. A two reach test canal has been considered and all proposed control laws have been implemented on the SIC complete nonlinear model developed at Cemagref \cite{56}.

For sake of completeness, some more synthesis procedures based on parameter optimization are also mentioned in Section 3.5, even if not applied to the case of interest.
3.2 Decentralized control by solving a linear least squares problem

Let us first present the new design procedure proposed by the author in [88].

3.2.1 Design procedure

Let us consider a linear time-invariant (LTI) plant model $\mathcal{P}$ with $n$ states and $l$ control channels given by

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{l} B_i u_i(t) \\
y_i(t) &= C_i x(t) \\
i &= 1, \ldots, l
\end{align*}
$$

(3.1)

where $x \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{r_i}$ are the inputs and outputs, respectively, of the $i$-th control station, and $A$, $B_i$ and $C_i$ are matrices of appropriate sizes.

Let $m = \sum_{i=1}^{l} m_i$, $r = \sum_{i=1}^{l} r_i$, and let (3.1) be written as follows:

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
$$

(3.2)

where

$$
B = \begin{bmatrix} B_1, B_2, \cdots, B_l \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_l \end{bmatrix}.
$$

The problem of determining a centralized output feedback control law of the form

$$
u(t) = K^* y(t)
$$

where $K^*$ is an unconstrained matrix such that a given performance index is minimized, or the desired closed-loop eigenstructure is assigned, can be quite easily solved even for great order systems [32, 46, 47, 71].

On the contrary, this is not the case if a block diagonal feedback gain matrix

$$
K = \text{block diag} \{ K_1, \cdots, K_l \}
$$

(3.3)

with $K_i \in \mathbb{R}^{m_i \times r_i}$, $i = 1, \ldots, l$ have to be determined.

The aim of the numerical procedure we propose is that of determining the feedback gain matrix $K$ so that the state evolution of the resulting decentralized closed-loop system

$$
\dot{x}(t) = (A + BK) x(t)
$$

(3.4)
approaches, in the least squares sense, the state evolution of the optimal centralized closed-loop model

\[ \dot{x}(t) = (A + BK^*C)x(t). \]  

(3.5)

At this purpose, let us rewrite system (3.4) as follows

\[ \dot{x}(t) = Ax(t) + \sum_{i=1}^{l} B_i K_i C_i x(t). \]  

(3.6)

By multiplying matrices \( B_i, K_i, C_i \) and the state vector \( x \), the previous equation can be rewritten as

\[ \dot{x}(t) = Ax(t) + \sum_{i=1}^{l} \begin{bmatrix} m_i \\ \vdots \\ m_i \end{bmatrix} \sum_{j_1=1}^{r_i} \sum_{j_2=1}^{n} B_i(1, j_1) K_i(j_1, j_2) C_i(j_2, j_3) x_{j_3}(t) \]  

(3.7)

Note that (3.7) is a system of ordinary differential equations, linear with respect to the unknown parameters \( K_i(j_1, j_2), j_1 = 1, \ldots, m_i, j_2 = 1, \ldots, r_i \).

The synthesis problem can then be turned into the identification of matrices \( K_i, i = 1, \ldots, l \), in the linear model (3.7). To solve this problem we need to know \( x(t) \) and \( \dot{x}(t) \) at a large enough number of time instants. But, as we will explain in the following, these values can be easily obtained.

At this purpose, an important remark should take place. Since we want the closed-loop decentralized system behaviour approaches the closed-loop centralized one for any initial state, data needed for the identification process are provided by computing \( n \) different evolutions of centralized system (3.5) in an appropriate time interval. Such evolutions should be estimated by considering as initial states the \( n \) column-vectors \( e_k, k = 1, \ldots, n \) of the \( n \) order identity matrix. As a result of the linearity of the system under study, all the other evolutions of system (3.5) can be expressed as a linear combination of the previous ones. Therefore, such a choice does not introduce any limitation. Henceforth, we will denote as \( x^{(k)}(t) \) the state evolution of system (3.5) corresponding to the initial state \( x(0) = e_k \).

Furthermore, we will denote as \([0, t_f]\) the time interval of interest, i.e., the time interval of observational data. The final time instant \( t_f \) is determined such that \( x^{(k)}(t_f) = 0 \) for all \( k = 1, \ldots, n \).

Within the fixed time interval, a discrete set of time instants \( \{t_0, t_1, \ldots, t_N\} \) has to be selected. The simplest choice consists in dividing it in \( N \) sub-intervals of equal amplitude. In such a case \( t_j = t_{j-1} + \delta, j = 1, \ldots, N \), with \( t_0 = 0 \) and \( \delta = t_f/N \). Therefore, the set of
observational data is given by $x^{(k)}(t_j), k = 1, \cdots, n, j = 0, \cdots, N$. At the same time instants, by virtue of (3.7), also $\dot{x}^{(k)}(t_j), k = 1, \cdots, n,$ can be computed.

Therefore, from system (3.7) the following linear over-determined system

$$\Delta \nu = \gamma$$

(3.8)
derives, where $\nu$ is the vector of unknown parameters defined as

$$\nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_l \end{bmatrix}, \quad \nu_i = \begin{bmatrix} K_i(1,1) \\ \vdots \\ K_i(1,r_i) \\ K_i(2,1) \\ \vdots \\ K_i(2,r_i) \\ \vdots \\ K_i(m_i,1) \\ \vdots \\ K_i(m_i,r_i) \end{bmatrix} \in \mathbb{R}^{m_i r_i}, \quad i = 1, \cdots, l,$$

$\gamma$ is the data vector

$$\gamma = \begin{bmatrix} \dot{x}^{(1)}(t_0) - Ax^{(1)}(t_0) \\ \vdots \\ \dot{x}^{(1)}(t_{N-1}) - Ax^{(1)}(t_{N-1}) \\ \dot{x}^{(2)}(t_0) - Ax^{(2)}(t_0) \\ \vdots \\ \dot{x}^{(2)}(t_{N-1}) - Ax^{(2)}(t_{N-1}) \\ \vdots \\ \dot{x}^{(n)}(t_0) - Ax^{(n)}(t_0) \\ \vdots \\ \dot{x}^{(n)}(t_{N-1}) - Ax^{(n)}(t_{N-1}) \end{bmatrix} \in \mathbb{R}^{n^2 N},$$

and $\Delta \in \mathbb{R}^{n^2 N \times \sum_{i=1}^l m_i r_i}$ is the matrix of coefficients whose entries immediately follow from system (3.7), taking into account the above definition of the unknown vector $\nu$.

System (3.8) can be solved in the least square sense, i.e., $\nu$ can be determined as the solution of the following linear least squares problem

$$\min_{\nu \in \mathbb{R}^{\sum_{i=1}^l m_i r_i}} \| \Delta \nu - \gamma \|_2.$$
Obviously, the solution of the above optimization problem does not ensure the stability of the resulting decentralized closed-loop system. It can only be verified a posteriori, as often happens in many other synthesis procedures in the field of decentralized control.

Theoretically the solution depends on the choice of both the time interval \([0,t_f]\) and the sampling points within it. However, in all the numerical examples considered, it has been possible to determine the smallest time interval such that the accuracy of the solution does not improve any more when its amplitude increases [88]. Analogously, it has also been observed that the solution is the same even if the number of sampling instants \(\{t_0,t_1,\cdots,t_N\}\) exceeds a certain value [88].

On the other hand, the actual procedure presents several advantages. In particular, it can be really effective even when dealing with great order problems [88]. In fact, the accuracy of the solution of a linear least squares problem depends on the ratio between the number of unknowns and the number of equations, which is proportional to the number of available data. But, in our case, an arbitrarily large set of data can be obtained by numerically solving system (3.5). Effective routines for this are provided by Matlab [58] (see \texttt{ode23} and \texttt{ode45}) and the computational complexity of the method is not excessive even when a large number of feedback gains should be evaluated. No problem of local minima and good initial estimate of the solution occurs. Furthermore, it is easy to introduce some constraints on the sign and/or the amplitude of the unknown parameters. In such a case, the unconstrained optimization problem (3.9) becomes a constrained one, but it can be solved with a modest additional computational effort.

Finally, the above procedure can be immediately adapted to discrete-time synthesis problems. Therefore, also discrete-time decentralized output feedback control laws can be designed by solving a linear least squares problem.

It is to note that the above procedure can also be used when differently structured feedback gain matrices have to be computed.

Obviously, the efficiency of this methodology highly depends on the choice of both an appropriate closed-loop target system and an appropriate structured feedback gain matrix.

### 3.2.2 A numerical application

The above procedure has been applied to a test canal, already considered in the literature [19, 21, 85]. It corresponds to the general scheme shown in figures 1.1–1.2 and has the following characteristics:

- length of the first reach: \(l_1 = 4000m\);
- length of the second reach: \(l_2 = 5000m\);
- canal bottom slope: \(p_1 = 0.0003\);
• water level depth in upstream reservoir measured from the canal bottom in the upper end section: \( h_M = 2.5 \text{ m} \);
• water level depth in downstream reservoir measured from the canal bottom in the lower end section: \( h_V = 1 \text{ m} \);
• trapezoidal cross section with \( w = 1.7 \text{ m}, \theta = 45^\circ \);
• constant opening section of the third gate: \( \sigma_3 = \sigma_{03} = 2.41 \text{ m}^2 \);
• discharge coefficient: \( \eta = 0.6 \);
• roughness coefficient: \( \gamma = 0.36 \).

The nominal configuration of uniform flow is characterized by the following levels and discharge values:

• water level depth in the 1st reach: \( h_{01} = 1.70 \text{ m} \);
• water level depth in the 2nd reach: \( h_{02} = 1.20 \text{ m} \);
• flow rate in the 1st reach: \( q_{01} = 5.94 \text{ m}^3/\text{s} \);
• flow rate in the 2nd reach: \( q_{02} = 3.02 \text{ m}^3/\text{s} \);
• user flow rate at the 1st reach lower end: \( q_{0c1} = 2.92 \text{ m}^3/\text{s} \);
• user flow rate at the 2nd reach lower end: \( q_{0c2} = 0.15 \text{ m}^3/\text{s} \);
• opening section of the 1st gate: \( \sigma_{01} = 2.50 \text{ m}^3/\text{s} \);
• opening section of the 2nd gate: \( \sigma_{02} = 1.61 \text{ m}^3/\text{s} \).

Now, let us consider as a target the centralized control law obtained via the LQR technique [46] applied to the linear-time invariant model of the open–channel, denoted as approximate nominal model. The unconstrained feedback gain matrix is obtained as the solution of the following linear optimization problem by solving a Riccati equation:

\[
\min J = \int_0^\infty [v(t)^TQv(t) + \sigma(t)^TR\sigma(t)]dt
\]
\[
s.t.
\]
\[
\dot{v}(t) = Av(t) + B\sigma(t)
\]

where

\[
A = \begin{bmatrix} -1.50 & 0.56 \\ 1.10 & -1.90 \end{bmatrix} \cdot 10^{-4}, \quad B = \begin{bmatrix} 1.81 & -0.92 \\ -0.33 & 1.31 \end{bmatrix},
\]

46
The structure of $Q$ has been chosen such that the volume variations in each reach, with respect to the initial volume, are weighted in the same manner. $R$ has been assumed to be scalar so as to control all the gates with the same energy. $R = 50000$ is an appropriate numerical value determined by a trial and error procedure.

Finally, it has been evaluated that an appropriate choice for the final time instant is $t_f = 500s$, and $N = 200$ is a sufficient number of sampling points within $[0, t_f]$.

As a result the following diagonal feedback gain matrix

$$K = -\begin{bmatrix} 5.4 & 0 \\ 0 & 3.8 \end{bmatrix} \cdot 10^{-3}$$

has been obtained.

In figure 3.2 we present a comparison between the behaviour of the system governed by centralized (dashed lines) and decentralized (continuous lines) control law in the presence of the unknown disturbances reported in fig 3.1.a.

As it can be noted, differences between the two state evolutions are quite negligible.
Figure 3.2: Comparison between percentage centralized (dashed lines) and decentralized (continuous lines) controlled system: (a) percentage volume variation in the first reach; (b) percentage volume variation in the second reach; (c) gate opening variation in the first reach; (d) gate opening variation in the second reach.
A larger dimensional problem has been studied and presented in [88]. Even in this case the results of numerical simulations revealed to be satisfactory.

3.3 Decentralized control via Edmunds’ algorithm

3.3.1 Edmunds’ algorithm

Suppose that our plant is represented by transfer function matrix $G$, with $l$ columns (inputs) and $m$ rows (outputs), and that we are to design a controller with transfer-function matrix $K$ (with $m$ columns and $l$ rows). Let the closed-loop transfer function actually achieved by a controller $K$ be $T = GK(I + GK)^{-1}$, and let the ‘target’ transfer function which we would like to achieve be $T_t$. Corresponding to $T_t$, there is a ‘target’ controller $K_t$ such that

$$GK_t = T_t(I - T_t)^{-1}. \tag{3.10}$$

We define an error function

$$E = T_t - T. \tag{3.11}$$

Then it can be shown, with a little manipulation, that

$$(I - T)(GK_t - GK)(I - T_t) = E. \tag{3.12}$$

If we suppose that $\|E\|$ is sufficiently small, which will be the case if $K$ is sufficiently close to $K_t$, then, by replacing $I - T$ by $I - T_t$ in (3.12), we obtain

$$(I - T_t)(GK_t - GK)(I - T_t) \approx E \tag{3.13}$$

since

$$(I - T)(GK_t - GK)(I - T_t) = (I - T_t)(GK_t - GK)(I - T_t) + O(\|E\|^2). \tag{3.14}$$

Now let us write

$$K(s) = \frac{1}{d(s)}N(s) \tag{3.15}$$

where $d(s)$ is a common-denominator polynomial which is assumed to be known, and $N(s)$ is a matrix of polynomials of known degrees but with unknown coefficients. Finally, we define

$$L(s) = I - T_t(s) \tag{3.16}$$

$$M(s) = \frac{1}{d(s)}L(s)G(s) \tag{3.17}$$

and

$$Y(s) = L(s)G(s)K_t(s)L(s). \tag{3.18}$$

Then (3.13) becomes

$$Y(s) \approx M(s)N(s)L(s) + E(s). \tag{3.19}$$
The noteworthy features here are that the unknown coefficients in $N(s)$ appear linearly in this expression, that $M(s)$, $L(s)$ and $Y(s)$ are well known and can be evaluated at particular values of $s$ when required and, hence, that the problem of finding $N(s)$ which minimizes

$$
\|E\|^2 = \int_{-\infty}^{\infty} tr[E^T(-j\omega)E(j\omega)]d\omega
$$

is a linear least-squares problem if the approximate equality in (3.19) is replaced by exact equality.

To put (3.19) into the more familiar standard form in which linear least-squares problems are usually seen, we need to ‘stack’ the columns of $Y$, $N$ and $E$ on top of each other. For this purpose we define their columns by

$$
Y(s) = \begin{bmatrix} y_1(s) & \cdots & y_m(s) \end{bmatrix} \tag{3.20}
$$

$$
N(s) = \begin{bmatrix} n_1(s) & \cdots & n_m(s) \end{bmatrix} \tag{3.21}
$$

$$
E(s) = \begin{bmatrix} e_1(s) & \cdots & e_m(s) \end{bmatrix} \tag{3.22}
$$

We also need to use the $\otimes$ notation for the Kronecker or tensor product of two matrices: if $\bar{P}$ has $p$ rows and $q$ columns, and $\bar{Q}$ has $r$ rows and $s$ columns, then $\bar{P} \otimes \bar{Q}$ is the $pr \times qs$ matrix:

$$
\bar{P} \otimes \bar{Q} = \begin{bmatrix}
p_{11}Q & p_{12}Q & \cdots & p_{1s}Q \\
p_{21}Q & p_{22}Q & \cdots & p_{2s}Q \\
\vdots & \vdots & & \vdots \\
p_{r1}Q & p_{r2}Q & \cdots & p_{rs}Q
\end{bmatrix} \tag{3.23}
$$

In this notation, (3.19) can be written as

$$
\begin{bmatrix} y_1(s) \\
y_2(s) \\
\vdots \\
y_m(s) \end{bmatrix} \approx \begin{bmatrix} L^T(s) \otimes M(s) \end{bmatrix} \begin{bmatrix} n_1(s) \\
n_2(s) \\
\vdots \\
n_m(s) \end{bmatrix} + \begin{bmatrix} e_1(s) \\
e_2(s) \\
\vdots \\
e_m(s) \end{bmatrix} \tag{3.24}
$$

Remember that $n_i(s)$ represents a vector of polynomials:

$$
n_i(s) = \begin{bmatrix} n_{i1}(s) & \cdots & n_{i1}(s) \end{bmatrix}^T \tag{3.25}
$$

and suppose that

$$
n_{ij}(s) = \nu_{ij}^0 s^p + \nu_{ij}^1 s^{p-1} + \cdots + \nu_{ij}^p \tag{3.26}
$$

for some positive integer $p$, assuming for notations convenience that each $\nu_{ij}$ has the same degree. This is not a real restriction, since $\nu_{ij}^p = 0$ is allowed, and can be forced if desired. Then $\{\nu_{ij}\}$ is the set of controller parameters to be optimized; if each $n_{ij}$ has degree $p$, there
are \( lm(p+1) \) of them. We need to introduce one more new notation: let \( \sigma(s) \) be the matrix (with \( lm \) rows and \( lm(p+1) \) columns)

\[
\Sigma(s) = \begin{bmatrix}
s^p & s^{p-1} & \cdots & 1 \\
0 & s^p & s^{p-1} & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & s^p & s^{p-1} & \cdots & 1
\end{bmatrix}
\]

then

\[
\begin{bmatrix}
n_1(s) \\
\vdots \\
n_m(s)
\end{bmatrix} = \Sigma(s)\nu
\]

where

\[
\nu = \begin{bmatrix}
\nu^0_{11} & \nu^1_{11} & \cdots & \nu^p_{ml}
\end{bmatrix}^T.
\]

so if we let

\[
\varrho(s) = \begin{bmatrix} L^T(s) \otimes M(s) \end{bmatrix} \Sigma(s)
\]

and

\[
\varepsilon(s) = \begin{bmatrix} \varepsilon^T_1(s) & \cdots & \varepsilon^T_m(s) \end{bmatrix}^T
\]

then (3.24) becomes

\[
\varrho(s) \approx X(s)\nu + \varepsilon(s)
\]

which is in a standard form; \( \varrho(s) \) is a known matrix, \( \nu \) is a vector of unknown parameters and \( \varepsilon(s) \) is a vector of ‘errors’.

To obtain a practical algorithm, we need to evaluate \( \varrho(s) \) and \( X(s) \) at a number of points on the imaginary axis, say \( \{s = j\omega_i : i = 1, 2, \cdots, \mu\} \), and approximate \( \|E\|_2 \) (which is the same as \( \|\varepsilon\|_2 \)) by

\[
\|E\|_2^2 \approx \sum_{i=1}^\mu \varepsilon^T(j\omega_i)\varepsilon(j\omega_i).
\]

Assembling data from all these points, we obtain

\[
\begin{bmatrix}
\varrho(j\omega_1) \\
\vdots \\
\varrho(j\omega_{\mu})
\end{bmatrix} \approx \begin{bmatrix} X(j\omega_1) \\
\vdots \\
 X(j\omega_{\mu})
\end{bmatrix} \nu + \begin{bmatrix} \varepsilon(j\omega_1) \\
\vdots \\
 \varepsilon(j\omega_{\mu})
\end{bmatrix}.
\]
The standard least-squares solution to this would be [48]:

\[
\hat{\nu} = \left\{ \begin{bmatrix} \mathbf{X}^T(-j\omega_1) & \cdots & \mathbf{X}^T(-j\omega_\mu) \end{bmatrix} \right\}^{-1} \begin{bmatrix} \mathbf{X}(j\omega_1) \\ \vdots \\ \mathbf{X}(j\omega_\mu) \\ \phi(j\omega_1) \\ \vdots \\ \phi(j\omega_\mu) \end{bmatrix}
\]

but in general this would give complex parameter values. We therefore have to depart slightly from the standard problem, in order to obtain real parameters. To do this, we use the following lemma:

**Lemma 1.** If \( \mathbf{Y} = \mathbf{X}\theta + \mathbf{E} \), the value of \( \theta \) which minimizes \( \|\mathbf{E}\|_2 \), given \( \mathbf{X} \) and \( \mathbf{Y} \), and subject to the constraint \( \text{Im}\{\theta\} = 0 \), is

\[
\hat{\theta} = \left[ \text{Re}\left\{\mathbf{X}^H \mathbf{X}\right\} \right]^{-1} \text{Re}\left\{\mathbf{X}^H \mathbf{Y}\right\}.
\]  

(3.37)

With the aid of this lemma we obtain the optimal real parameters:

\[
\hat{\nu} = \left\{ \begin{bmatrix} \text{Re}\left\{\mathbf{X}^T(-j\omega_1) \cdots \mathbf{X}^T(-j\omega_\mu)\right\} \end{bmatrix} \right\}^{-1} \begin{bmatrix} \text{Re}\left\{\mathbf{X}(j\omega_1) \right\} \\ \vdots \\ \text{Re}\left\{\mathbf{X}(j\omega_\mu) \right\} \\ \text{Re}\left\{\phi(j\omega_1) \right\} \\ \vdots \\ \text{Re}\left\{\phi(j\omega_\mu) \right\} \end{bmatrix}
\]

(3.38)

The validity of this algorithm depends on the validity of (3.13), which in turn depends on the size of \( \|\mathbf{E}\|_2 \) obtained with the parameter vector \( \hat{\nu} \).

It is imperative that the actual algorithm employed solves (3.35) by using a numerically stable procedure and does not use (3.38), since in the neighborhood of the true solution the matrix

\[
\text{Re}\left\{ \begin{bmatrix} \mathbf{X}^T(-j\omega_1) & \cdots & \mathbf{X}^T(-j\omega_\mu) \end{bmatrix} \right\}
\]

approaches singularity. A numerically stable algorithm is obtained as follows. In the notation of Lemma 7.1, let \( \mathbf{X} = \mathbf{X}_{\text{Re}} + j\mathbf{X}_{\text{Im}}, \mathbf{Y} = \mathbf{Y}_{\text{Re}} + j\mathbf{Y}_{\text{Im}} \), then \( \hat{\theta} \), defined by (3.37), is also obtained as the least squares solution of the equation

\[
\begin{bmatrix} \mathbf{Y}_{\text{Re}} \\ \mathbf{Y}_{\text{Im}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{\text{Re}} \\ \mathbf{X}_{\text{Im}} \end{bmatrix} \theta + \begin{bmatrix} \mathbf{E}_{\text{Re}} \\ \mathbf{E}_{\text{Im}} \end{bmatrix}.
\]
This can be solved in a numerically stable way by using the Householder transformation and QR factorization [95].

3.3.2 Decentralized control law

Now, let us assume that a proportional decentralized control law has to be designed, i.e., we want to determine a diagonal matrix \( K \) such that
\[
T(s) = G(s)K(I + G(s)K)^{-1}
\]
approaches, in the frequency range of interest, the target closed–loop function \( T_t(s) \).

At this purpose, we assume
\[
d(s) = 1
\]
and
\[
\Sigma(s) = \begin{bmatrix}
e_1' & 0 & 0 & \cdots & 0 \\
0 & e_2' & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e_n'
\end{bmatrix} \in \mathbb{R}^{n^2 \times n},
\]
where \( e_i \) is the \( i \)-th row of the \( n \) order identity matrix and \( 0 \) is an \( n \) order vector whose entries are all equal to zero.

In such a way, all the off–diagonal elements of the feedback gain matrix are constrained to zero.

3.3.3 PI decentralized control law

Now, let us show how a PI decentralized control law can be easily designed. Thus, we have to determine a diagonal gain matrix \( K \) such that
\[
T(s) = G(s)K(s)(I + G(s)K(s))^{-1}
\]
approaches, in the frequency range of interest, the target closed–loop function \( T_t(s) \), with each non–null element of \( K(s) \) consisting of a linear combination of a real constant and a term proportional to \( 1/s \).

At this purpose, we assume
\[
d(s) = s
\]
and
\[
\Sigma(s) = \begin{bmatrix}
e_1' & se_1' & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & e_2' & se_2' & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & e_n' & se_n'
\end{bmatrix} \in \mathbb{R}^{n^2 \times 2n},
\]
where $e_i$ is the $i$-th row of the $n$ order identity matrix and $0$ is an $n$ order vector whose entries are all equal to zero.

### 3.3.4 A numerical application

In this subsection we show how the Edmund’s algorithm can be satisfactorily applied to design both a proportional and a proportional integral decentralized controller when dealing with two reach test canal already considered in section 3.2.2.

As already specified above, the effectiveness of the Edmund’s algorithm depends on the choice of the target closed loop function which should be compatible with both the model and the feedback gain matrix structure. At this purpose, the target closed loop matrix function $T_t$ has been designed by means of the LQR technique [46] applied to the linear-time invariant model of the open–channel, denoted as approximate nominal model. Therefore, the feedback gain matrix $K_t$ has been obtained as the solution of the following linear optimization problem by solving a Riccati equation:

$$
\begin{align*}
\min J &= \int_0^{\infty} [v(t)^T Q v(t) + \sigma(t)^T R \sigma(t)] dt \\
\text{s.t.} & \\
\dot{v}(t) &= A v(t) + B \sigma(t)
\end{align*}
$$

where matrices $A$, $B$, $Q$ and $R$ are the same as those already considered in section 3.2.2. Thus we obtained

$$K_t = \begin{bmatrix} -4.31 & -1.12 \\ 0.91 & -4.74 \end{bmatrix} \cdot 10^{-3}$$

and

$$T_t(s) = G_A(s)K_t(I + G_A(s)K_t)^{-1}$$

where $G_A(s) = (sI - A)^{-1}B$.

Now, we want to design a proportional decentralized control law

$$\sigma(t) = K v(t)$$

by applying the Edmund’s algorithm. Therefore we assumed $d(s) = 1$ and

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In such a way all the off-diagonal elements of the feedback gain matrix are constrained to zero.

Since the most important hydraulic phenomena occur in a low frequency range, we assumed $w_1 = 10^{-9}$ and $w_\mu = 10^{-2}$. We further evaluated that a good choice for $\mu$ is 250.
In such a way the diagonal gain matrix is

$$K = \begin{bmatrix} -6.91 & 0 \\ 0 & -4.42 \end{bmatrix} \cdot 10^{-3},$$

while the corresponding closed loop transfer matrix is

$$T(s) = G(s)K(I + G(s)K)^{-1},$$

where $G(s)$ is the nonrational transfer matrix defining the *reference model* of the open–channel.

To evaluate the effectiveness of the applied procedure we compared the Bode diagrams of all the elements of $T$ and $T_t$. The results of such a comparison are reported in figure 3.3. As it can be clearly seen, the Bode diagrams are really close even in a wider frequency range than that considered for the controller parameter identification procedure.

In the following we also present the results of a numerical simulation that demonstrate the good performance of the system in the presence of unknown disturbances, when the above
decentralized control law is implemented. Note that all numerical simulations have been carried out on the commercial SIC software [56].

The unknown disturbances are those given in figure 3.1.a. The results of simulations are shown in figure 3.4: a) shows the volume percentage variations; the gate opening variations are shown in b). As it can be seen, the system’s behaviour is rather satisfactory, but a steady state error still holds.

As it is well known, steady state performance can be improved by means of an integral action in the feedback loop. The above described procedure can easily be extended for designing a PI decentralized controller. As in the previous case, the target closed loop function is determined by taking into account the linear time–invariant model. At this purpose, the following augmented state equation:

\[
\begin{align*}
\dot{v}(t) &= A v(t) + B \sigma(t) \\
\dot{\psi}(t) &= v(t)
\end{align*}
\]

has been considered. It can be rewritten in the more compact form:

\[
\dot{v}(t) = \tilde{A} \tilde{v}(t) + \tilde{B} \sigma(t)
\]
where
\[ \tilde{v}(t) = \begin{bmatrix} v(t) \\ \bar{v}(t) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ I & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \]

\( I \) is the \( n \) order identity matrix and \( 0 \) the null \( n \) order matrix. Now, the \( n \times 2n \) feedback gain matrix \( \tilde{K} \) can be determined by solving the following optimization problem:
\[
\min J = \int_0^\infty [\tilde{v}(t)^T \tilde{Q} \tilde{v}(t) + \sigma(t)^T R \sigma(t)] dt \\
\text{s.t.} \quad \dot{\tilde{v}}(t) = \tilde{A} \tilde{v}(t) + \tilde{B} \sigma(t).
\]
Matrices \( A, B \) and \( R \) are the same as those in the previous case, while \( \tilde{Q} \) has been chosen as:
\[
\tilde{Q} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & v_{10}/v_{20} & 0 & 0 \\
0 & 0 & 5 \cdot 10^{-6} & 0 \\
0 & 0 & 0 & 5 \cdot 10^{-6}
\end{bmatrix},
\]
i.e., weighting of the first two components of the state and of the control variables are the same as those already used in the simpler proportional decentralized controller case. The other numerical values have been obtained as the result of a trial and error procedure.

In this way, we obtained that the optimal feedback control law for the augmented linear system is
\[
\sigma(t) = \begin{bmatrix}
-5.55 & -2.19 \\
0.99 & -6.13
\end{bmatrix} \cdot 10^{-3} \cdot v(t) + \begin{bmatrix}
-0.96 & -0.27 \\
0.27 & -0.96
\end{bmatrix} \cdot 10^{-5} \cdot \bar{v}(t).
\]
Therefore, the \( n \times n \) feedback gain matrix \( K_t(s) \) is equal to
\[
K_t(s) = \begin{bmatrix}
-5.55 \cdot 10^{-3} - 0.96/s \cdot 10^{-5} & -2.19 \cdot 10^{-3} - 0.27/s \cdot 10^{-5} \\
0.99 \cdot 10^{-3} + 0.27/s \cdot 10^{-5} & -6.13 \cdot 10^{-3} - 0.96/s \cdot 10^{-5}
\end{bmatrix},
\]
while \( T_t(s) \) is defined as above.

Now, the controller parameter identification procedure can be applied as in the simpler proportional case with only two differences: it has been assumed \( d(s) = s \) and
\[
\Sigma(s) = \begin{bmatrix}
s & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & s & 1
\end{bmatrix}.
\]
The frequency range considered for parameter identification is the same as above. In such a way the diagonal gain matrix is
\[
K(s) = \begin{bmatrix}
-4.40 \cdot 10^{-3} - 1.30/s \cdot 10^{-5} & 0 \\
0 & -3.80 \cdot 10^{-3} - 0.92/s \cdot 10^{-5}
\end{bmatrix}.
\]
Figure 3.5: Bode diagrams of all the elements of $T$ and $T_t$ in the case of proportional integral control law.
As in the proportional decentralized case, a comparison between the Bode diagrams of all the elements of $T$ and $T_t$ is presented. As it can be seen in figure 3.5, the diagonal elements of the two transfer function matrices are practically coincident, while at low frequency range significant differences exist between the off diagonal elements. However, it is important to note that these elements are negligible in magnitude with respect to the others.

To demonstrate the effectiveness of the integral action with respect to the steady state error, we also present the results of a further numerical simulation. The second simulation test case has been performed under the assumption that the unknown disturbances are those shown in figure 3.1.b while the results of simulation are reported in figure 3.6.

### 3.4 Decentralized control via $H_2$ norm minimization

#### 3.4.1 Conditions on $H_2$ norm

A standard approach to the control of linear time–invariant multiple–input multiple–output systems considers a block diagram such as that of figure 3.7 (see [24, 28]). In this figure $P(s)$ is the transfer matrix of the generalized plant, while $K(s)$ is the transfer matrix of the controller.
The vector $w$ represents all external inputs, such as disturbances, sensor noise and reference signals, while the vector $y$ is an error signal. The vector $r$ is the set of observed variables used by the controller to compute the control input $u$. The closed loop transfer matrix between $w$ and $y$ is called lower linear fractional transformation (LFT) of $P$ and $K$ and is denoted $F_l(P, K)$.

Let us consider the linear model of a system to be controlled

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
z(t) &= Cx(t)
\end{align*}$$

(3.39)

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, and $z \in \mathbb{R}^p$ is the output vector.

Classical LQR problem formulation [46] requires to find the state feedback law $u(t) = Kx(t)$, with $K \in \mathbb{R}^{m \times n}$, such that the cost functional

$$J = \int_0^\infty [z^T(t)Qz(t) + u^T(t)Ru(t)]dt$$

(3.40)

is minimized for any initial state $x(0) = x_0$. Here $Q = Q^T \geq 0$, $R = R^T > 0$. The solution to the LQR problem is:

$$K^* = -R^{-1}B^TX,$$

(3.41)

where $X$ is the solution of the algebraic Riccati equation

$$XA + A^TX - XBR^{-1}B^TX + C^TQC = 0.$$  

The closed loop poles are the eigenvalues of $A + BK^*$.

The equivalent frequency domain problem [24] is to find the state feedback matrix $K^*$
such that the norm $\|F_l(P, K)\|_2$ is minimized \(^2\), where the transfer matrix of the generalized plant $P(s)$ has the following expression in terms of state space data:

$$P(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \begin{bmatrix} x_p \\ w \\ u \end{bmatrix}.$$ 

The state space equation of the generalized plant is

$$\dot{x}_p(t) = Ax_p(t) + Bu(t) + w(t),$$

and from (3.43) it can be seen that whenever $x(t) = x_p(t)$

$$\|y\|_2^2 = J.$$ 

Since

$$y(s) = F_l(P, K)w(s),$$

it is possible to prove [24] that by minimizing the norm $\|F_l(P, K)\|_2$ Eq. (3.43) is minimized for any external input of the form $w(t) = x_0\delta(t)$.

### 3.4.2 Decentralized control law

To design a proportional decentralized control law, a diagonal structure to the feedback gain matrix has been imposed:

$$K_d = \text{diag} \{k_1, k_2, \ldots, k_N\}.$$ 

\(^2\)Let $g(t) : \mathbb{R} \to \mathbb{R}^{m \times n}$ be a signal matrix and $g(s)$ its Laplace transform. The $H_2$ norm of $g$ is:

$$\|g\|_2 = \left( \int_{-\infty}^{\infty} \text{trace} \{g^T(t)g(t)\} dt \right)^{1/2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \{g^H(j\omega)g(j\omega)\} d\omega \right)^{1/2}.$$
We want to determine the $K_d^*$ matrix whose control law is the best approximation of
\[ u(t) = K^* x(t) \] (3.45)
where $K^*$ is the optimal (unconstrained) matrix. Thus we want to determine the $N$ parameters, $k_1, k_2, \cdots, k_N$ which minimize $\| F_l(P, K_d^*) \|_2$. Clearly,
\[ \| F_l(P, K_d^*) \|_2 \geq \| F_l(P, K^*) \|_2 \] (3.46)
since the right hand side of equation (3.46) is a global minimum. The decentralized law performance will in general be worse than those given by (3.45).

Let us discuss the physical significance of $H_2$ norm minimization. If the generalized plant is excited with $N$ different disturbances $w_i(t) = e_i \delta(t)$, where $e_i$ is the $i$-th canonical basis vector, denoting with $y_i(t)$ the corresponding error signal, then
\[ \sum_{i=1}^N \| y_i \|_2 = \| F_l(P, K_d^*) \|_2. \] (3.47)
Since the $\| y_i \|_2^2$ can be considered as the value $J_{d,i}$ of the performance index (4.3) when the decentralized system starts from the initial condition $x(0) = e_i$, by minimizing the $\| F_l(P, K_d^*) \|_2$ we also minimize the $\sum_{i=1}^N J_{d,i}$ among all possible decentralized systems.

The decentralized system does not possess the basic property of optimal control, namely that of minimizing the performance index (4.3) for any initial state $x(0)$. It is possible, however to find upper bounds for the value $J_d$ taken by (4.3) when the feedback matrix is $K_d^*$. Let $y(s)$ and $y_d(s)$ be the outputs of the generalized plant in the case of centralized and decentralized control when the input is $w(s) = x_0$. Then
\[ y(s) = F_l(P, K^*)x_0 \]
\[ y_d(s) = F_l(P, K_d^*)x_0. \] (3.48)
It is possible to prove [23] that the performance index of the centralized system is bounded by
\[ J = \| y \|_2^2 \leq \| F_l(P, K^*) \|_2^2 \| x_0 \|_2^2, \] (3.49)
while the performance index of the decentralized system is bounded by
\[ J_d = \| y_d \|_2^2 \leq \| F_l(P, K_d^*) \|_2^2 \| x_0 \|_2^2. \] (3.50)
Note that in the previous equations $\| x_0 \|_2$ is the Euclidean norm of a vector, while the norm $\| F_l(P, K_d^*) \|_2$ is a transfer function norm. These equations have an interesting physical interpretation. They show that the value of the $H_2$ norm is an upper bound for the value of the performance index under arbitrary initial conditions on the unitary sphere. When the numerical values of $\| F_l(P, K^*) \|_2$ and $\| F_l(P, K_d^*) \|_2$ are close, we may conclude that for any arbitrary initial conditions the performance indexes $J$ and $J_d$ have close upper bounds.

Physically, the higher value of the performance index $J_d$ is due to the fact that the decentralized system’s response is slower. In fact, in the centralized control scheme each control input has immediate knowledge of the system’s state, while in the decentralized control scheme this information is transmitted by the perturbation propagation.
3.4.3 PI decentralized control law

The above described procedure, as the previous one, can easily be extended for the design of a PI decentralized controller. In this case the problem is that of determining the \(2n\) gain factors \(k_{Pi}, \ i = 1, \cdots, n\), such that the \(n\) control laws:

\[
u_i(t) = -k_{Pi}x_i(t) - k_{Ii} \int_0^t x_i(\tau)d\tau, \quad i = 1, \cdots, n
\]  

are optimal according to a chosen performance index.

For this purpose, let us consider the augmented system state equation:

\[
\begin{aligned}
    \dot{x}(t) &= Ax(t) + Bu(t) \\
    \dot{\bar{x}}(t) &= x(t)
\end{aligned}
\]  

(3.52)

Which can be rewritten in a more compact form as follows:

\[
\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t)
\]  

(3.53)

where

\[
\tilde{x}(t) = \begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ I & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},
\]

\(I\) is the \(n\) order identity matrix and \(0\) the null \(n\) order matrix. Now, we can apply the \(H_2\) norm minimization procedure described above. In the case at hand the \(2n \times n\) feedback gain matrix has the following structure:

\[
\tilde{K_d} = \begin{bmatrix} k_{P_1} & 0 & 0 & k_{I_1} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & k_{P_n} & 0 & 0 & k_{I_n} \end{bmatrix}.
\]  

(3.54)

3.4.4 A numerical application

Let us consider again the two reach canal whose geometrical and physical characteristics are given in section 3.2.2.

To design a proportional decentralized controller we assumed as weighting matrices the same as in section 3.2.2. To determine the optimal parameters of the diagonal feedback gain matrix \(K_d^*\) we have to solve the following optimization problem

\[
\min_{k_1, k_2} \| F_l(P, K_d) \|_2.
\]

For this purpose we used the software tools available in Matlab: \texttt{fmins} is the minimization procedure and \texttt{normh2} compute the \(H_2\) norm. The optimal values computed are: \(k_1^* = -7 \cdot 10^{-3}\), \(k_2^* = -7 \cdot 10^{-3}\).
$k_2^* = -6 \cdot 10^{-3}$. These values give $\| F_l(P, K^*_d) \|_2 = 366$. While the optimal $K^*$ matrix is:

\[
K^* = \begin{bmatrix}
-4.3 & -1.1 \\
0.9 & -4.7
\end{bmatrix} \cdot 10^{-3}
\]

and gives $\| F_l(P, K^*) \|_2 = 359$.

We can observe that in the case at hand the difference between $\| F_l(P, K^*_d) \|_2$ and $\| F_l(P, K^*) \|_2$ is of the order of 2%, i.e., it is negligible.

In the following we present the results of some numerical simulations carried out on the commercial SIC software.

In the first simulation the unknown disturbances are those reported in figure 3.1.a. A comparison between centralized and decentralized control law is also presented. Figures 3.8.a and 3.8.b show the volume percentage variations, while the gate opening variations are shown in figure 3.8.c and 3.8.d. We can observe that in the case examined decentralized control law performs better than centralized control law. This is not in conflict with the theoretical results discussed above. In fact, all the theoretical results have been obtained for the nominal model, while simulations have been performed on a completely unsteady non-linear model. In the case at hand, where unknown disturbances occur in both canal reaches, decentralized control law guarantees better performance since the diagonal elements in the feedback gain matrix are higher in magnitude and disturbances occur simultaneously in both canal reaches, thus there is no time lag.

Figure 3.9 shows the results of a second simulation test case performed assuming that a disturbance occurs only in the second canal reach. The shape and magnitude of the disturbance is the same as in the previous simulation. Graphical results are in accordance with the conclusion drawn above: decentralized control law ensures better performance in the reach where unknown disturbances occur and worse performance in the other.

However, in all cases examined the differences between the corresponding curves are not significant. This is a consequence of the small difference between $\| F_l(P, K^*_d) \|_2$ and $\| F_l(P, K^*) \|_2$.

Now, let us examine the case of a PI decentralized controller. For this purpose, let us assume that the weighting matrices associated to the augmented system are those already considered in the previous chapter.

Now, to determine the optimal parameters we need to find the

\[
\min_{k_{P1}, k_{P2}, k_{I1}, k_{I2}} \| F_l(\tilde{P}, \tilde{K}_d) \|_2
\]

where $\tilde{P}$ is the transfer matrix of the generalized plant associated to the augmented system (3.52).
Figure 3.8: Comparison between percentage centralized (dashed lines) and decentralized (continuous lines) controlled system in the case of the first simulation: (a) percentage volume variation in the first reach; (b) percentage volume variation in the second reach; (c) gate opening variation in the first reach; (d) gate opening variation in the second reach.
Figure 3.9: Comparison between percentage centralized (dashed lines) and decentralized (continuous lines) controlled system in the case of the second simulation: (a) percentage volume variation in the first reach; (b) percentage volume variation in the second reach; (c) gate opening variation in the first reach; (d) gate opening variation in the second reach.
The optimal values obtained are: 

\[ k_{P_1}^* = -5.5 \cdot 10^{-3}, \quad k_{P_2}^* = -6.1 \cdot 10^{-3}, \quad k_{I_1}^* = -1.18 \cdot 10^{-5}, \quad k_{P_1}^* = -8.67 \cdot 10^{-6}. \]

These values give \( \| F_l(\tilde{P}, \tilde{K}_d) \|_2 = 733 \) while the optimal unconstrained \( \tilde{K}^* \) matrix gives \( \| F_l(P, \tilde{K}^*) \|_2 = 483 \). It is to note that in this case the difference between the two norms is much relevant. Even so, to illustrate the effectiveness of the integral action in addition to the proportional one, the unknown disturbances reported in figure 3.1.b have been introduced. Results are shown in figure 3.10: a) shows the volume percentage variations, while changes in gate openings are reported in b). As it can be seen, even in this case satisfactory results have been obtained.

An important remark needs to be done. Note that finding the optimal decentralized control law is a problem of optimization. As such, it is almost certainly not a convex optimization problem and there may be multiple solutions that locally minimize the norm 2 of the lower linear fractional transformation. However, in the case at hand, starting from different initial values of the parameters to identify, we observed that the minimization procedure always converges to the same value of these parameters.

### 3.5 Other methodologies

In this section we want to mention two more approaches to the problem: the method of inequalities and the multi-objective optimization. They are both capable of solving very general design problems, and are not inherently restricted to the frequency domain. Both can be implemented quite easily, using standard optimization subroutines. Their effectiveness depends, however, on good user interfaces, and the development of these is an area of current research.

#### 3.5.1 The method of inequalities

If one observes operators controlling a process manually, it becomes apparent that in many cases their objective is to hold a number of plant variables between permitted bounds, and that they are indifferent to the detailed behaviour of the variables, provided they remain within those bounds. The design specifications can therefore be written as a set of inequalities:

\[
|e_i(t)| \leq \varepsilon_i, \quad t \geq 0, \quad i = 1, 2, \ldots, m
\]  

(3.55)

where each \( e_i(t) \) is some function of time which represents the behaviour of one plant variable.

Zakian [105, 106] was apparently the first to suggest that many feedback–control design problems should be posed as the satisfaction of a set of inequalities, rather than the minimization of some objective function with inequalities acting as side–constraints. The shift of emphasis from an objective function to a set of inequalities gives a more accurate formal representation of many design problems, and leads to an iterative design procedure in which the
Figure 3.10: The results of the third simulation: (a) percentage volume variations; (b) gate opening variations.
designer changes the 'trade–off' between conflicting constraints by adjusting the inequalities, rather than some objective function. This is attractive, because it is usually much easier to understand the physical implications of changes in constraining inequalities than of changes in an objective function.

Optimization is still used in Zakian’s approach, as will become clear shortly, but the shift of emphasis from minimization to satisfaction means that the usual ideas on choice of optimization algorithms have to be revised: speed of convergence in the neighborhood of a minimum becomes much less important than the likelihood of finding at least one feasible point – namely one at which all the inequalities are satisfied, and the extent of information given to the user about causes of failure – which can be used to modify the inequalities, thus making success more likely at the next design iteration.

Inequalities constraints arise from physical limitations as well as from desired objectives. For example, magnitude and/or rates of change of control signals are usually limited, which results in a set of inequalities of the form

\[ |u_i(t)| \leq \eta_i, \quad |\dot{u}_i(t)| \leq \nu_i. \]  

(3.56)

Similar inequalities can arise not from real physical limits, but from the fact that one’s model of the plant is valid only for certain ranges of variables.

If \( p \) is a vector of controller parameters which are to be designed, with a prespecified controller structure (e.g. a decentralized structure), then inequalities (3.55) or (3.56) assume the form

\[ \psi_i(p) \leq 0 \]  

(3.57)

where each \( \psi_i(p) \) is a functional. For example, to get (3.55) we define

\[ \psi_i(p) = \max_{0 \leq \tau < \infty} |e_i(t, p)| - \varepsilon_i \]  

(3.58)

where \( e_i(t, p) \) now denotes the function \( e_i(t) \) which is obtained with controller parameter \( p \). If we wish to impose closed–loop stability as a requirement, we can define

\[ \psi_i(p) = \max \{ \text{Re\{closed–loop poles obtained with } p \} \} + \varepsilon \]  

(3.59)

for some positive real \( \varepsilon \). To constrain the value of one of the parameters (to avoid excessive gain) we use a simple function of \( p \):

\[ \psi_i(p) = |p_i| - p_{\text{max}} \]  

(3.60)

and so on.

Very many design objectives and constraints can be put in the form (3.57), even though it may not be straightforward to express \( \psi_i(p) \) mathematically. In particular, if we have an ‘envelope constraint’, such as

\[ a(t) \leq y(t, p) \leq b(t) \]  

(3.61)
where \(a(t)\) and \(b(t)\) are functions of time, we can represent it by (3.57) if we define

\[ \psi_i(p) = \max_t \max_{\delta \in \Delta} [a(t) - y(t, p), y(t, p) - b(t)]. \] (3.62)

This is referred to as an ‘infinite–dimensional constraint’, since it can be regarded as the ‘limit’ of the set of inequalities

\[
\begin{align*}
  a(t_1) &\leq y(t_1, p) \leq b(t_1) \\
  \vdots \\
  a(t_N) &\leq y(t_N, p) \leq b(t_N)
\end{align*}
\] (3.63)

as \(N\) is increased indefinitely. Frequency–domain constraints can also be represented by (3.57). Problems of robust design can be represented similarly: suppose that the variable \(y\) depends on a plant parameter vector \(\delta\) as well as the controller parameter vector \(p\), and we have a ’robust performance’ requirement that

\[
a(t) \leq y(t, p, \delta) \leq b(t), \quad \forall \delta \in \Delta. \] (3.64)

Then we define

\[ \psi_i(p, \delta) = \max_t \max_{\delta \in \Delta} \max_{0 \leq t < \infty} [a(t) - y(t, p, \delta), y(t, p, \delta) - b(t)]. \] (3.65)

Let \(S_i\) be the set of parameter vectors for which the \(i\)-th functional inequality is satisfied:

\[ S_i = \{p : \psi_i(p) \leq 0\}. \] (3.66)

Then the admissible or feasible set of parameters vectors, for which all the inequalities hold, is the intersection

\[ S = \bigcap_i S_i. \] (3.67)

Clearly \(p\) is an admissible parameter vector if

\[ \max_i \psi_i(p) \leq 0 \] (3.68)

which shows that the search for an admissible \(p\) can be pursued by optimization, in particular by solving

\[ \min_p \max_i \psi_i(p). \] (3.69)

Different algorithms have been proposed for the solution of the above minimization problem. In particular, we remember [38, 59, 60, 98, 107].

### 3.5.2 Multi–objective optimization

A formulation of the control–system design problem, which has close similarities to the method of inequalities, has been proposed by Kresisselmeier and Steinhauser in [44, 45]. Instead of a
set of inequalities, they translate a specific design problem into a set of performance objectives \( \{ J_i(p), i = 1, 2, \ldots, m \} \), which are chosen such that if \( J_i(p_1) < J_i(p_2) \) then the design parameter (vector) \( p_1 \) is better than the design parameter \( p_2 \), as far as the objective \( J_i \) is concerned. This set of objectives can be considered to be a vector of objective functions, or a multi-objective function:

\[
J(p) = \begin{bmatrix}
J_1(p) \\
\vdots \\
J_m(p)
\end{bmatrix}.
\]

(3.70)

The performance specification is satisfied if \( J(p) < c \) for some positive vector \( c \) of 'thresholds' (i.e. if \( J_i(p) < c_i \) for each \( i \), and \( c_i > 0 \)). At this stage the formulation can be seen to be completely equivalent to that of the method of inequalities. As in that case, the appropriate values of some of the elements of \( c \) are usually not known in advance, since they depend on qualitative judgements about trade-offs between competing objectives. These are dealt with by initially being given large values, so that the corresponding objectives play little or no role in the initial stage of design. As the design progresses the values of these elements of \( c \) are reduced until they begin to affect the design. The idea is that if this is done in a suitable interactive environment then the user can discover what compromises are necessary, and can adjust the elements of \( c \) accordingly.

### 3.6 Conclusions

The idea that optimization of parameters within a fixed structure should be used to design control systems is by no means new. It was applied as early as the 1950s, but the state of contemporary computing technology prevented its widespread exploitation. Today the position is completely different, since with the actual computer and workstations, results of optimization can be obtained within a few seconds or minutes, already displayed graphically and annotated. Secondly, the powerful optimization algorithms have been developed. This allows success even with difficult problems and enlarge the class of allowable performance functionals, as well as having an effect on speed. Thirdly, and perhaps, most importantly, is the increasing exploitation of the modern computing environment. Work is already under way on the development of integrated environments which make it easy for the casual or inexperienced user to connect software for simulation and optimization and thus formulate a particular design problem, without needing to be an expert in either simulation or optimization – and without being an expert programmer, either.

In this chapter we have mainly presented three relatively straightforward methods of optimizing the parameters of a controller for a feedback system, assuming that its structure has been already chosen.

Firstly, a really simple procedure for the design of a feedback structured controller for linear time invariant systems has been presented. It is inspired by previous research work of
the author in the field of identification. It reduces the synthesis process to the identification of a set of unknown parameters in a linear system of ordinary differential equations. Despite of its simplicity, it presents the advantage with respect to much more sophisticated methodologies, to maintain quite the same computational complexity even if the dimension of the problem increases.

The Edmunds’ method solves a specific frequency–domain optimization problem, using a rather crude algorithm. Nevertheless, when used intelligently it is capable of finding very good controllers of relatively low complexity.

Finally, a new methodology for the design of structured control laws has been proposed. Such a methodology is based on the minimization of the $H_2$ norm of an appropriate transfer function matrix and is the equivalent in the frequency domain, of the LQR technique in the time domain. It has been satisfactorily applied to the problem of interest. The only drawback of this methodology is that it requires the solution of a nonlinear optimization problem, thus difficulties related to local minima may occur, especially when dealing with large sets of unknown parameters. However, this problem can be partially overcome by imposing some a priori relationships between unknown parameters, i.e., by taking into account the physical knowledge of the system at hand, thus reducing the number of unknowns.

In all cases, the results of some numerical simulations have also been presented.

Finally, for sake of completeness, at the end of this chapter we mention some more procedures that enables us to study the problem of designing a decentralized control law as a problem of parameter optimization.
Chapter 4

Decentralized control laws via eigenstructure assignment

4.1 Introduction

The problem of eigenstructure assignment (simultaneous assignment of eigenvalues and eigenvectors) is of great importance in control theory and applications because the stability and dynamic behaviour of a linear multivariable system are governed by the eigenstructure of the system. In general, the speed of the dynamic response of a linear system depends on its eigenvalues whereas the "relative shape" of the dynamic response depends on the associated eigenvectors. Eigenstructure assignment by state feedback control and output feedback control have received considerable attention over the last decade. Eigenstructure assignment has been used successfully in incorporating desirable control requirements, such as robustness of eigenvalue assignment [65] and least gain controller [69], into eigenvalue assignment by state feedback control or by output feedback control.

State feedback control or output feedback control requires that every controller have access to all the measurements taken from the system. For large scale systems, this requirement presents a need for long–distance communication that can be impractical and uneconomical. To circumvent the difficulty, decentralized control is widely used to control large scale systems such as electric power systems [49] and large space structures [37]. Despite its importance, however, eigenstructure assignment by decentralized feedback control is an unsolved problem.

Over the past decade, many methods have been proposed for eigenvalue (pole) assignment by decentralized control [16, 66, 102]. However, it is not easy to incorporate eigenvector assignment into the process of eigenvalue assignment using these methods.

In this chapter we present an approach firstly proposed by Lu et al. in [52] to eigenstructure assignment by decentralized control that revealed to be really effective when applied to
the physical system of interest [93, 94]. We derive a parametric expression for the set of feedback gains of decentralized controllers which achieve the desired eigenvalue assignment. The free parameters in this parametric expression can be used to assign eigenvalues and eigenvectors simultaneously. This new approach is an extension of the existing methods that achieve eigenstructure assignment by either state feedback control or output feedback control.

As in the previous chapters, the results of some numerical simulations relative to the case of interest are finally presented.

4.2 Design procedure

In this section we recall the fundamental steps of an efficient method proposed by Lu et al. [52] to design a decentralized control law via eigenstructure assignment.

Consider a linear time-invariant system described in state space

\[ \dot{x}(t) = Ax(t) + \sum_{i=1}^\nu B_i u_i(t), \quad y_i(t) = C_i x(t), \quad i = 1, \ldots, \nu \]  

(4.1)

where \( x(t) \in \mathbb{R}^n \) is the state, \( u_i(t) \in \mathbb{R}^{m_i}, \ y_i(t) \in \mathbb{R}^{r_i} \) are the input and output respectively, of the \( i \)-th local control station. Matrices \( A, B_i \) and \( C_i, i = 1, \ldots, \nu \) are real, constant and of appropriate size.

If a decentralized feedback control law \( u_i(t) = F_i y_i(t), \ F_i \in \mathbb{R}^{m_i \times r_i}, i = 1, \ldots, \nu \) is applied to system (4.1), a closed-loop system is obtained in the form

\[ \dot{x}(t) = \left( A + \sum_{i=1}^\nu B_i F_i C_i \right) x(t) = (A + BFC) x(t) \]  

(4.2)

where

\[ B = \begin{bmatrix} B_1 & \cdots & B_\nu \end{bmatrix} \in \mathbb{R}^{n \times m}, \quad m = \sum_{i=1}^\nu m_i \]  

(4.3)

\[ C = \begin{bmatrix} C_1 \\ \vdots \\ C_\nu \end{bmatrix} \in \mathbb{R}^{r \times n}, \quad r = \sum_{i=1}^\nu r_i \]  

(4.4)

\[ F = \text{block diag } \left[ F_1, \ \cdots \ F_\nu \right] \in \mathbb{R}^{m \times r}. \]  

(4.5)

Let us define the set of decentralized feedback gains by

\[ F = \{ F = \text{block diag } \left[ F_1, \ \cdots \ F_\nu \right] \in \mathbb{R}^{m \times r} \mid F_i \in \mathbb{R}^{m_i \times r_i} \}. \]  

(4.6)

Lu et al. in [52] provide a valid procedure to determine a decentralized feedback gain matrix \( F \in F \) so that
1. *(Eigenvalue Assignment):* System (4.2) is assigned an arbitrary self-conjugate set of \( k \) eigenvalues \( \{ \mu_i, i = 1, \cdots, k \} \), \( k \leq n \), namely

\[
\{ \mu_i, i = 1, \cdots, k \} \subset \Lambda(A + BFC) = \Lambda \left( A + \sum_{i=1}^{n} B_i F_i C_i \right)
\] (4.7)

where \( \Lambda(\cdot) \) denotes the set of eigenvalues.

2. *(Eigenvector Assignment):* \( v_i, i = 1, \cdots k \), the closed-loop eigenvectors (with unit norm) associated with \( \{ \mu_i, i = 1, \cdots, k \} \), minimize certain performance indexes.

One commonly used performance index for assigning eigenvectors \( v_i, i = 1, \cdots, k \) is

\[
J = \sum_{i=1}^{k} (v_i - v_i^d)^T W (v_i - v_i^d)
\] (4.8)

where \( W = \text{diag}(w_1, \cdots, w_m) \) is a weighting matrix, \( \{ v_i^d, i = 1, \cdots, k \} \) is a set of desired unit normed eigenvectors which reflect our requirement on the shape of the closed-loop dynamic response. In general, it is not possible to exactly assign \( v_i \) to \( v_i^d \) and minimization of (4.8) gives the eigenvectors closest to the desired ones [52].

**4.2.1 Eigenvalue assignment**

In this subsection, we recall several analytical results firstly proposed and demonstrated by Lu et al. in [52], to characterize the decentralized feedback gain matrices that achieve the desired eigenvalue assignment for linear system (4.1). In the following we assume that linear system (4.1) is a controllable and observable system.

**Theorem 2.** *(Characterization of Decentralized Control for Eigenvalue Assignment):* Let \( \{ \mu_i, i = 1, \cdots, k \} \) be a self-conjugate set of distinct complex numbers such that

\[
\{ \mu_i, i = 1, \cdots, k \} \cap \Lambda(A) = \emptyset.
\]

There exists a decentralized feedback gain \( F \subset F \) such that the eigenvalues of the closed-loop system (4.2) contain \( \{ \mu_i, i = 1, \cdots, k \} \) if and only if there exist non-null parameter vectors \( p_i \in \mathbb{C}^m, i = 1, \cdots, k \), satisfying the following two conditions:

1. \( p_i \in \mathbb{R}^m \) if \( \mu_i \) is real, and \( p_i = p_i^* \in \mathbb{C}^m \) if \( \mu_i = \mu_j^* \);

2. \[
[I - FH(\mu_i)] p_i = 0, \quad i = 1, \cdots, k
\] (4.9)

where \( H(s) = C(sI - A)^{-1}B \) is the open-loop transfer function.
Proof. (If): Let a decentralized feedback gain $F \in \mathcal{F}$ assign eigenvalues $\{\mu_i, i = 1, \ldots, k\}$ to matrix $A + BFC$, and $v_i$ be the eigenvector associated with $\mu_i$, then

$$(\mu_i I - A - BFC)v_i = 0$$

(4.10)

which leads to

$$(\mu_i I - A)v_i = BFCv_i.$$ 

(4.11)

We will show that $p_i = FCv_i$, $i - 1, \ldots, k$ satisfy conditions 1) and 2). Condition 1) holds obviously. Substituting $p_i = FCv_i$ into (4.11), we have

$$v_i = (\mu_i I - A)^{-1}Bp_i.$$ 

(4.12)

Therefore,

$$p_i = FCv_i = FC(\mu_i I - A)^{-1}Bp_i = FH(\mu_i)p_i.$$ 

(4.13)

This is condition 2) of the theorem statement.

(Only if): Let $p_i$, $i = 1, \ldots, k$ satisfy conditions 1) and 2) of the theorem statement. Then

$$p_i = FH(\mu_i)p_i = FC(\mu_i I - A)^{-1}Bp_i.$$ 

(4.14)

Let

$$v_i = (\mu_i I - A)^{-1}Bp_i$$

(4.15)

then,

$$p_i = FCv_i.$$ 

(4.16)

Since $(A, B)$ is controllable, from (4.15), the nonzeroness of $p_i$ implies the nonzeroness of $v_i$. From (4.15) and (4.16), we have

$$(\mu_i I - A - BFC)v_i = B(p_i - FCv_i) = 0$$

(4.17)

therefore, $\mu_i, i = 1, \cdots, k$ are the eigenvalues of the closed-loop system and $v_i, i = 1, \cdots, k$ are the associated eigenvectors. 

Note that results similar to Theorem 2 exist in literature, for example, see [68].

Furthermore, let us observe that, if vectors $p_i \in \mathbb{C}^m$, $i = 1, \cdots, k$ satisfy the conditions stated in Theorem 2, then the closed-loop eigenvectors associated with $\{\mu_i, i = 1, \cdots, k\}$ are

$$v_i = (\mu_i I - A)^{-1}Bp_i, \ i = 1, \cdots, k.$$ 

(4.18)

The following theorem concerns the existence of a solution to (4.9). However, it is first necessary to prove an important lemma.
Lemma 3. Assume that the eigenvalues of $A$ are distinct. Let $\{\lambda_i, i = 1, \cdots, k\} \subset \Lambda(A)$ and $v_i, w_i, i = 1, \cdots, k$ be the corresponding right and left eigenvectors of $A$. Define a matrix $T$ by

$$T = \begin{bmatrix} (C_1v_1)^T \otimes (w_1^TB_1) & \cdots & (C_1v_\nu)^T \otimes (w_1^TB_\nu) \\ \vdots & \ddots & \vdots \\ (C_kv_k)^T \otimes (w_k^TB_1) & \cdots & (C_kv_k)^T \otimes (w_k^TB_\nu) \end{bmatrix}$$

(4.19)

where $M \otimes V$ is the Kronecker product of $M$ and $V$.

Let $f(\mu_i)$ be a $r$-dimensional vector of bounded functions of $\mu_i$ in the neighborhood of $\lambda_i$, namely, there exists a small number $\varepsilon > 0$ such that

$$\|f(\mu_i)\| < M_i, \ \forall \mu_i \in \overline{D}_i \equiv \{\mu : |\mu - \lambda_i| \leq \varepsilon\}$$

(4.20)

where $M_i$ is a constant. If rank$(T) = k$, then there exists a small number $\varepsilon > 0$ such that for all $\mu_i \in D_i \equiv \{\mu : |\mu - \lambda_i| \leq \varepsilon\}$, we have a solution $F$ to the following set of equations

$$\overline{B}_iF(\overline{C}_1 + (\mu_i - \lambda_i)f(\mu_i)) = \mu_i - \lambda_i, \ i = 1, \cdots, k$$

(4.21)

where $\overline{B}_i = w_i^TB, \overline{C}_i = Cv_i$.

Proof. Recall that $F$ has the form $F = \text{block diag}(F_1, \cdots, F_\nu)$. Define $\text{col}(F)$ by

$$\text{col}(F) = \begin{bmatrix} \text{col}(F_1) \\ \vdots \\ \text{col}(F_\nu) \end{bmatrix}$$

(4.22)

where $\text{col}(F_i)$ is a vector formed by stacking the columns of $F_i$ from left to right. By algebraic manipulation, the set of (4.21) is equivalent to

$$(T + \overline{T})\text{col}(F) = \begin{bmatrix} \mu_1 - \lambda_1 \\ \vdots \\ \mu_k - \lambda_k \end{bmatrix}$$

(4.23)

where $T$ is as in (4.19) and $\overline{T}$ is expressed as

$$\overline{T} = \begin{bmatrix} \mu_1 - \lambda_1 & 0 & \cdots & 0 \\ 0 & \mu_2 - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_k - \lambda_k \end{bmatrix}$$

(4.24)
where \( f_j(\mu_i) \) is the part of \( r \)-vector function \( f(\mu_j) \) corresponding to \( C_j v_i \). Obviously, for any small \( \delta > 0 \), there exists an \( \varepsilon > 0 \) such that \( |\mu_i - \lambda_i| < \varepsilon, i = 1, \ldots, k \) implies \( \|T\| < \delta \). As a result, if \( T \) is of full rank, there exists an \( \varepsilon > 0 \) such that \( |\mu_i - \lambda_i| < \varepsilon, i = 1, \ldots, k \) implies that \( T + T \) is of full rank, therefore (4.23) has a solution.

**Theorem 4.** (Existence of Decentralized Control): Assume that the eigenvalues of \( A \) are distinct. Let \( \{\lambda_i, i = 1, \ldots, k\} \subset \Lambda(A) \) and \( v_i, w_i, i = 1, \ldots, k \) be the corresponding right and left eigenvectors of \( A \). Define a matrix \( T \) as in the above lemma.

If \( \text{rank}(T) = k \), then there exists an \( \varepsilon > 0 \) such that for any set \( \{\mu_i, i = 1, \ldots, k\} \) satisfying

\[
\mu_i \in D_i \equiv \{\mu : 0 < |\mu - \lambda_i| \leq \varepsilon\}, \quad i = 1, \ldots, k
\]

there exists a solution \( F \in \mathcal{F} \) to

\[
[I - FH(\mu_i)] p_i = 0, \quad i = 1, \ldots, k
\]  (4.25)

where \( p_i \neq 0, i = 1, \ldots, k \).

**Proof.** Since the eigenvalues of \( A \) are distinct, we have

\[
(\mu_i I - A)^{-1} = T(\mu_i I - \Lambda)^{-1}T^{-1}
\]  (4.26)

where \( \Lambda = \text{diag}(\lambda_1 \cdots \lambda_n) \), and \( T = [v_1 \cdots v_n], T^{-1} = [w_1 \cdots w_n]^T \). Therefore,

\[
H(\mu_i) = C(\mu_i I - A)^{-1}B = CT(\mu_i I - \Lambda)^{-1}T^{-1}B
\]

\[
= \sum_{j=1}^{n} \frac{1}{\mu_i - \lambda_j} \overline{C}_j \overline{B}_j, \quad i = 1, \ldots, k
\]  (4.27)

where \( \overline{C}_j = C_j v_j, \overline{B}_j = w_j^T B \).

Let

\[
g(\mu_i) = \sum_{j=1}^{n} \frac{1}{\mu_i - \lambda_j} \overline{C}_j \overline{B}_j,
\]

then using (4.27), (4.25) is equivalent to

\[
F(\overline{C}_i \overline{B}_i + (\mu_i - \lambda_i)g(\mu_i)) p_i = (\mu_i - \lambda_i) p_i, \quad i = 1, \ldots, k.
\]  (4.28)

Form a matrix \( U \in \mathcal{C}^{m \times m} \) as follows:

\[
U = \begin{bmatrix} \overline{B}_i \\ V \end{bmatrix}
\]  (4.29)
where \( V \in \mathbb{C}^{(m-1) \times m} \) is a matrix such that \( U \) is nonsingular. Let \( \overline{p}_i = Up_i \) and \( h(\mu_i) = g(\mu_i)U^{-1} \), then (4.28) becomes

\[
UF(C_iB_iU^{-1} + (\mu_i - \lambda_i)h(\mu_i))\overline{p}_i = \left[
\begin{array}{cccc}
\overline{B}_iF(C_i + (\mu_i - \lambda_i)h_{11}(\mu_i)) & \cdots & * \\
VF(C_i + (\mu_i - \lambda_i)\overline{h}_1(\mu_i)) & \cdots & *
\end{array}
\right] \overline{p}_i
\]

(4.30)

where \( h_{11}(\mu_i) \) is the (1,1) element of matrix \( H(\mu_i) \), and \( [h_{11}(\mu_i) \overline{h}_1(\mu_i)]^T \) is the first column of \( H(\mu_i) \), * stands for the elements of the matrix irrelevant to the discussion.

From Lemma 3, we know that there exists a solution \( F \) to the following equations (we can check that the conditions of Lemma 3 are satisfied):

\[
\overline{B}_iF(C_i + (\mu_i - \lambda_i)h_{11}(\mu_i)) = \mu_i - \lambda_i, \quad i = 1, \cdots, k.
\]

(4.31)

Let \( F^* \) be the solution, and substitute it into (4.30). Since \( \overline{B}_iF^*(C_i + (\mu_i - \lambda_i)h_{11}(\mu_i)) = \mu_i - \lambda_i \neq 0 \), using Gaussian elimination, we can find a matrix \( Z \) such that

\[
Z \left[
\begin{array}{cccc}
\overline{B}_iF^*(C_i + (\mu_i - \lambda_i)h_{11}(\mu_i)) & \cdots & * \\
VF^*(C_i + (\mu_i - \lambda_i)\overline{h}_1(\mu_i)) & \cdots & *
\end{array}
\right] Z^{-1} = \left[
\begin{array}{cccc}
\overline{B}_iF^*(C_i + (\mu_i - \lambda_i)h_{11}(\mu_i)) & \cdots & * \\
0 & \cdots & *
\end{array}
\right].
\]

(4.32)

Now, it is easy to see that \( F^* \) and \( \overline{p}_i^* = \) the first column of \( Z^{-1} \) are a solution to (4.30). As a result, \( F^* \) and \( p_i^* = U^{-1}\overline{p}_i^* \) are a solution to (4.28), hence a solution to (4.25).

By taking into account the above theorems, Lu et al. [52] provide a parametric expression for a decentralized feedback gain matrix.

The following notations are useful in stating the next theorem:

\[
H_i(s) = C_i(sI - A)^{-1}B \in \mathbb{C}^{r_i \times m}
\]

\[
I_{m \times m} = \begin{bmatrix} E_1 & \{m_1 \\ E_2 & \{m_2 \\ \vdots & \vdots \\ E_{\nu} & \{m_{\nu} \end{bmatrix}, \quad m = \sum_{i=1}^{\nu} m_i
\]

\[
M^+ = \begin{cases} 
M^T(MM^T)^{-1}, & \text{if } M \in \mathbb{C}^{p \times q}, p < q \\
(M^T)^{-1}M^T, & \text{if } M \in \mathbb{C}^{p \times q}, p \geq q
\end{cases}
\]

79
\[ M^\perp = \begin{cases} I - M^T(MM^T)^{-1}M, & \text{if } M \in \mathbb{C}^{p \times q}, p < q \\ 0 \in \mathbb{C}^{q \times q}, & \text{if } M \in \mathbb{C}^{p \times q}, p \geq q \end{cases} \]

where \( I_{m \times m} \) is an \( m \) dimensional identity matrix.

**Theorem 5.** *(A Parametric Expression for Decentralized Control):*

Let \( \{\mu_i = 1, \cdots, k\} \) be a self-conjugate set of distinct complex numbers such that

\[ \{\mu_i, \ i = 1, \cdots, k\} \cap \Lambda(A) = \emptyset. \]

Then the eigenvalues of the closed-loop system (4.2) contains \( \{\mu_i = 1, \cdots, k\} \) if

\[ F_i = E_i[p_1, \cdots, p_k][H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k]^+, \quad i = 1, \cdots, \nu \quad (4.33) \]

where \( p_i, i = 1, \cdots, k \) satisfy

1. \( p_i \in \mathbb{R}^m \) if \( \mu_i \) is real, and \( p_i = p_i^* \in \mathbb{R}^m \) if \( \mu_i = \mu_i^* \);

2. matrix \( [p_1, \cdots, p_k][H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k] \in \mathbb{C}^{r_i \times k} \) is of full rank, \( i = 1, \cdots, \nu \);

3. \[ E_i[p_1, \cdots, p_k][H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k]^\perp = 0, \quad i = 1, \cdots, \mu \quad (4.34) \]

\[ \|p_j\| = 1, \quad j = 1, \cdots, k, \quad (4.35) \]

where \( \|\cdot\| \) is the Euclidean norm.

**Proof.** From (4.34) and the definition of \( [\ ]^\perp \), we have

\[ E_i[p_1, \cdots, p_k] = E_i[p_1, \cdots, p_k][H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k]^+ [H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k]. \quad (4.36) \]

Substituting the parametric expression (4.33) into (4.36), we have

\[ F_i[H_i(\mu_1)p_1, \cdots, H_i(\mu_k)p_k] = E_i[p_1, \cdots, p_k], \quad i = 1, \cdots, \nu \quad (4.37) \]

which implies

\[ F[H(\mu_1)p_1, \cdots, H(\mu_k)p_k] = [p_1, \cdots, p_k] \quad (4.38) \]

or

\[ FH(\mu_i)p_i = p_i, \quad i = 1, \cdots, k. \quad (4.39) \]

Therefore, by Theorem 2, the closed-loop matrix \( A + \sum_{i=1}^{\nu} B_i F_i C_i \) have eigenvalues \( \{\mu_i, \ i = 1, \cdots, k\} \).

Two main remarks can be done. First of all, the vector \( p_i \) explicitly parameterizes the closed-loop right eigenvector \( v_i \) associated with \( \mu_i \) according to

\[ v_i = (\mu_i I - A)^{-1} B p_i. \quad (4.40) \]
Furthermore, equation (4.33) gives a parametric expression of the desired feedback gain $F_i$, where the parameter vectors $p_i \in \mathbb{C}^m$, $i = 1, \ldots, k$, satisfy matrix nonlinear equations (4.34) and (4.35). If $r_i \geq k$, then $[H_i(\mu_1)p_1, \ldots, H_i(\mu_k)p_k]^\perp = 0$, therefore, (4.34) is trivially satisfied for any set of $p_i$ with $\|p_i\| = 1$. On the other hand, if $r_i < k$, then the matrix nonlinear equation (4.34) gives $m_i k$ nonlinear equations corresponding to this $r_i$. In order to obtain a set of $p_i$, $i = 1, \ldots, k$ satisfying (4.34), one needs to solve at most $\sum_{i=1}^\nu m_i k$ nonlinear equations. At this purpose, standard methods for solving nonlinear equations, such as Newton’s method can be used.

4.2.2 Eigenstructure assignment

The expression (4.33) provides a parametric expression for a class of decentralized feedback gains that achieve the desired eigenvalue assignment. In order to find such a feedback gain, we need to determine $p_i$, $i = 1, \ldots, k$ satisfying conditions 1), 2) and 3) stated in Theorem 3. In many cases, $p_i$, $i = 1, \ldots, k$ satisfying those conditions are nonunique. This freedom allows us to achieve eigenvector assignment in addition to eigenvalue assignment. For example, we may consider the following eigenstructure problem: find the parameter vectors $p_i$, $i = 1, \ldots, k$ which achieve the following minimization

$$
\min_{p_i, i=1,\ldots,k} \sum_{i=1}^k (v_i(p_i) - v_i^d)^T W (v_i(p_i) - v_i^d) \quad (4.41)
$$

subject to

$$
E_i[p_1, \ldots, p_k][H_i(\mu_1)p_1, \ldots, H_i(\mu_k)p_k]^\perp = 0, \quad i = 1, \ldots, \nu \quad (4.42)
$$

$$
\|v_i(p_i)\| = 1, \quad i = 1, \ldots, k \quad (4.43)
$$

where $v_i = (\mu_i I - A)^{-1} B p_i$.

The above minimization problem attempts to assign eigenvectors as close to the desired ones as possible while achieving required eigenvalue assignment as guaranteed by constraints (4.42) and (4.43). Constraint (4.43) ensures the unit norm of eigenvectors $v_i$, $i = 1, \ldots, k$. It is equivalent to constraint (4.35) in Theorem 3 in the sense that it ensures the nonzeroness of eigenvectors $v_i$, $i = 1, \ldots, k$.

When the minimization problem is solved, the required decentralized feedback gain matrix is obtained by substituting the resulting $p_i$, $i = 1, \ldots, k$ into formula (4.33).

4.3 A numerical application

In this section we show how the above method can be applied to design a proportional decentralized state feedback control law, i.e., how it is possible to determine a diagonal feedback
gain matrix $F$ such that the closed-loop behaviour of the approximate nominal model is characterized by the desired set of eigenvalues and the eigenvectors are as close as possible to the desired ones.

As in the previous chapters numerical simulations have been carried out on the test canal introduced in section 3.2.2.

Note that, since we want to design a state feedback control law, matrix $C$ has been chosen equal to the two order identity matrix.

In our numerical example, in accordance with the above notation, we have $n = m = r = \nu = 2, m_i = r_i = 1, i = 1, 2$. That is, we assume that the control actions are the gate opening variations, being each one a function of the only volume variation in the downstream gate, assumed to be measurable.

First of all, it has been verified that the system is controllable and the rank condition is verified.

Furthermore, a satisfactory closed-loop dynamic has been imposed. In particular we have assumed as a target the centralized control law obtained by means of the LQR technique in [19], i.e., it has been assumed that the performance index to be minimized is

$$J = \int_0^{\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

where, as in the previous cases,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & v_{10}/v_{20} \end{bmatrix}, \quad R = 50000.$$

The resulting closed-loop behaviour has been assumed as a target, i.e., we assumed that the desired eigenstructure is that obtained when the above mentioned centralized control law is applied. It is easy to verify that this is equivalent to impose:

$$\begin{bmatrix} v^d_1 \\ v^d_2 \end{bmatrix} = \begin{bmatrix} -0.82 \\ 0.57 \end{bmatrix}, \quad \begin{bmatrix} v^d_1 \\ v^d_2 \end{bmatrix} = \begin{bmatrix} -0.65 \\ -0.76 \end{bmatrix}, \quad \mu_1 = -1.08 \cdot 10^{-2}, \quad \mu_2 = -0.45 \cdot 10^{-2}.$$ 

Furthermore, we chose $W = I$ in equation (2.19) as we want to equally penalize the two distances between the desired and the actual eigenvectors.

In our numerical example, the constrained minimization problem (2.19-2.20-2.21) has been solved by means of the routine constr of MATLAB [58]. Such a routine requires an initial estimate of the solution. It has been evaluated that good initial values for $p_i, i = 1, 2$ can be obtained by assuming $v_i = v^d_i, i = 1, 2$ in equation (2.18).

In such a way, satisfactory results have been obtained: the eigenvalues are exactly coincident with the desired ones. On the contrary, an error on the first eigenvector occurs. In
\[ v_1 = \begin{bmatrix} -0.96 \\ 0.28 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -0.66 \\ -0.74 \end{bmatrix}. \]

In such a way, the following diagonal feedback gain matrix has been obtained
\[
F = \begin{bmatrix} -5.70 & 0 \\ 0 & -4.90 \end{bmatrix} \cdot 10^{-3}.
\]

To demonstrate the good performance of the system when the above decentralized control law is applied, we show the results of two different simulation test cases.

In the first one the linear stationary model has been considered for simulation. Figures 4.1-4.2 show the results of a comparison between the behaviour of the system when the unknown disturbances reported in figure 3.1.a occur and both the centralized and the decentralized control laws are implemented. Figure 4.1 shows the volume percentage variations in the two reaches, while figure 4.2 shows the gate opening variations, i.e., the two control actions. As it can be seen, differences are quite negligible. Such a simulation enabled us to test the real effectiveness of the eigenstructure assignment procedure under the assumption that no error on system modeling occurs.

However, from a physical point of view, it is much more interesting to repeat the same comparison when the two proportional control laws, the centralized and the decentralized one, are implemented on a completely nonlinear model, such as the SIC commercial software. In fact, such a model takes into account all the high frequency phenomena and the unsteady flow conditions neglected in the deduction of the linear time-invariant model. For this purpose, we considered once again the unknown disturbances reported in figure 3.1.a, while the results of the numerical simulation are shown in figures 4.3-4.4. Even in this case differences are quite negligible, i.e., decentralization does not imply significant loose of performance. Moreover, in the case at hand, where unknown disturbances occur in both canal reaches, decentralized control law guarantees better performance being the diagonal elements in the feedback gain matrix higher in magnitude, and disturbances occur simultaneously in both canal reaches, thus there is no time lag. Further simulations, not reported here for brevity’s sake, show that decentralized control law always ensures better performance in the reach where unknown disturbances occur and worse performance in the others.

As usual, the steady state error, even if not too relevant, can be eliminated by means of an integral action. For this purpose, we show how the above procedure can be satisfactorily applied even to design a PI decentralized controller. The only difference with respect to the previous case, is that now the following augmented state equation:
\[
\begin{align*}
\dot{v}(t) &= Av(t) + B\sigma(t) \\
\dot{\bar{v}}(t) &= v(t)
\end{align*}
\]
Figure 4.1: Comparison between percentage volume variations in the case of $P$ centralized (dashed line) and $P$ decentralized (continuous line) control when simulations are carried out on the linear model.
Figure 4.2: Comparison between gate opening variations in the case of P centralized (dashed line) and P decentralized (continuous line) control when simulations are carried out on the linear model.
Figure 4.3: *Comparison between percentage volume variations in the case of P centralized (dashed line) and P decentralized (continuous line) control when simulations are carried out on the nonlinear model.*
Figure 4.4: Comparison between gate opening variations in the case of P centralized (dashed line) and P decentralized (continuous line) control when simulations are carried out on the nonlinear model.
have to be considered. As usual, it can be rewritten in the more compact form:

\[ \dot{\tilde{v}}(t) = \tilde{A}\tilde{v}(t) + \tilde{B}\sigma(t) \]

where

\[
\tilde{v}(t) = \begin{bmatrix} v(t) \\ \bar{v}(t) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},
\]

\( I \) is the 2 order identity matrix and \( 0 \) the 2 order zero matrix.

Furthermore,

\[
\tilde{y}_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tilde{v}(t), \quad \tilde{y}_2(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tilde{v}(t),
\]

Therefore

\[
\sigma_1(t) = \tilde{F}_1(1,1)v_1(t) + \tilde{F}_1(1,2)\bar{v}_1(t) = \tilde{F}_1(1,1)v_1(t) + \tilde{F}_1(1,2) \int_{0}^{t} v_1(\tau)d\tau,
\]

\[
\sigma_2(t) = \tilde{F}_2(1,1)v_2(t) + \tilde{F}_2(1,2)\bar{v}_2(t) = \tilde{F}_2(1,1)v_2(t) + \tilde{F}_2(1,2) \int_{0}^{t} v_2(\tau)d\tau,
\]

In this case, in accordance with the above notation, we have \( n = r = 4, m = \nu = 2, m_i = 1, r_i = 2, i = 1, \cdots, 2. \)

Furthermore, it is easy to verify that the augmented system is controllable and observable, and the rank condition is verified.

As in the simpler proportional case, the desired closed-loop dynamic has been obtained by evaluating the optimal state feedback control law, solution of the following optimization problem:

\[
\min J = \int_{0}^{\infty} [\tilde{v}(t)^TQ\tilde{v}(t) + \sigma(t)^TR\sigma(t)]dt
\]

s.t.

\[ \dot{\tilde{v}}(t) = \tilde{A}\tilde{v}(t) + \tilde{B}\sigma(t) \]

where \( A, B \) and \( R \) are the same as above, while

\[
\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{10}/v_{20} & 0 & 0 \\ 0 & 0 & 5 \cdot 10^{-6} & 0 \\ 0 & 0 & 0 & 5 \cdot 10^{-6} \end{bmatrix}.
\]

As usual, weightings pertaining to the first two state components are the same as in the proportional case, while those pertaining to the latter two state components are the result of a trial and error procedure.
As in the proportional case, the centralized closed-loop behaviour has been assumed as a target, i.e., the desired eigenstructure is that obtained when the above mentioned centralized control law is applied to the approximate nominal model. This is equivalent to impose:

\[
v_1^d = \begin{bmatrix} -0.87 \\ 0.60 \\ 82.16 \\ -57.00 \end{bmatrix} \cdot 10^{-2}, \quad v_2^d = \begin{bmatrix} -0.27 \\ -0.27 \\ 70.80 \\ 70.62 \end{bmatrix} \cdot 10^{-2},
\]

\[
v_3^d = \begin{bmatrix} 0.22 \\ 0.13 \\ -86.13 \\ -50.82 \end{bmatrix} \cdot 10^{-2}, \quad v_4^d = \begin{bmatrix} 0.06 \\ 0.20 \\ 30.69 \\ -95.17 \end{bmatrix} \cdot 10^{-2},
\]

\[
\mu_1 = -1.05 \cdot 10^{-2}, \quad \mu_2 = -0.38 \cdot 10^{-2}, \quad \mu_3 = -0.26 \cdot 10^{-2}, \quad \mu_4 = -0.21 \cdot 10^{-2}.
\]

Furthermore, we have assumed \( W \) equal to the four order identity matrix in equation (2.19) as we want to equally penalize all the distances between the desired and the actual eigenvectors.

The optimization problem has been solved with the same routine of Matlab as above and the same procedure has been adopted to obtain an initializing estimate of the solution.

In such a way, satisfactory results have been obtained, even if the computational complexity of the problem, due to its higher dimension, did not allow us to exactly suit the desired eigenstructure. In fact, the resulting set of closed-loop decentralized eigenvalues is the following:

\[\{-1.07 \cdot 10^{-2}, -0.31 \cdot 10^{-2}, -0.15 \cdot 10^{-2}, -0.17 \cdot 10^{-2}\},\]

while the corresponding eigenvectors are:

\[
v_1 = \begin{bmatrix} -1.03 \\ 0.26 \\ 96.96 \\ -24.45 \end{bmatrix} \cdot 10^{-2}, \quad v_2 = \begin{bmatrix} -0.20 \\ -0.24 \\ 65.22 \\ 75.81 \end{bmatrix} \cdot 10^{-2},
\]

\[
v_3 = \begin{bmatrix} -0.15 \\ 0.04 \\ 96.66 \\ -25.63 \end{bmatrix} \cdot 10^{-2}, \quad v_4 = \begin{bmatrix} 0.00 \\ -0.07 \\ -5.43 \\ -99.85 \end{bmatrix} \cdot 10^{-2}.
\]

Finally, the block diagonal output feedback gain matrix is

\[
\tilde{F} = -\begin{bmatrix} 6.60 \cdot 10^{-3} & 8.29 \cdot 10^{-5} & 0 & 0 \\ 0 & 0 & 3.30 \cdot 10^{-3} & 2.04 \cdot 10^{-6} \end{bmatrix}.
\]
Figure 4.5: Comparison between percentage volume variations in the case of PI centralized (dashed line) and PI decentralized (continuous line) control when simulations are carried out on the nonlinear model.

To illustrate the effectiveness of the integral action in addition to the proportional one, the above law has been implemented in presence of the unknown step disturbances reported in figure 3.1.b. Figures 4.5-4.6 show the results of the comparison between the target centralized closed-loop system and the decentralized one. Once again, it can be observed that differences are quite negligible.

4.4 Conclusions

Eigenstructure assignment in linear multivariable systems is an important problem in linear system theory and has been studied by many authors in recent years. Up to present, attention in this aspect has been mainly concentrated on the cases of state feedback, output feedback and the case of dynamical compensators.

In this chapter an effective procedure for the design of a decentralized control law via eigenstructure assignment has been presented. A parametric expression for the set of feedback gains of decentralized controllers, which achieve the desired eigenvalue assignment, is derived.
Figure 4.6: Comparison between gate opening variations in the case of PI centralized (dashed line) and PI decentralized (continuous line) control when simulations are carried out on the nonlinear model.
The free parameters in this parametric expression can be used to assign eigenvalues and eigenvectors simultaneously. This new approach is an extension of the existing methods that achieve eigenstructure assignment by either state feedback control or output feedback control.

As revealed by numerical simulations, such an approach produces really satisfactory results when applied to the physical system under study.
Chapter 5

A decentralized volume variation observer

5.1 Introduction

In the previous chapters we introduced state feedback under the assumption that all state variables are available as outputs. This assumption often does not hold in practice, however, either because the state variables are not accessible for direct measurement or because the number of measuring devices is limited. Thus, in order to apply state feedback to stabilize or to optimize a system, a reasonable substitute for the state vector often has to be found. In this section we shall show how the available inputs and outputs of a dynamical equation can be used to drive a device so that the outputs of the device will approximate the state vector. The device that constructs an approximation of the state vector is called a state estimator or a state observer.

In this chapter we first provide some background on state estimators for linear time-invariant systems.

Then we show how this theory can be applied to the case of interest. In particular, the linear decoupled dynamical model of open-channels formulated in Chapter 2, enables us the design of decentralized volume variation observers.

The results of some numerical simulations are also presented. They have been performed both in presence of initial error on the volume variation estimates and of unknown disturbances acting on the canal reaches.
5.2 Background on asymptotic state observers

In this section we recall the main steps in the deduction of a full–order dynamical state observer for linear time–invariant systems. For more details on this subject we address the reader to [14] where also the case of reduced–order state observer is dealt. In the following, we limit ourselves to the discussion of the first type of state–estimators, being the only ones of interest for our application.

Consider the $n$–dimensional linear time–invariant dynamical equation

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) 
\end{align}

(5.1)

where $A$, $B$ and $C$ are, respectively, $n \times n$, $n \times p$ and $q \times n$ real constant matrices. For simplicity, the direct transmission part has been assumed to be zero. We further assume that the state variables are not accessible. Note that although the state variables are not accessible, the matrices $A$, $B$ and $C$ are assumed to be completely known.

Consider the state estimator shown in Figure 5.1. The estimator is driven by the input
as well as the output of the original system. The output of (5.1), $y = C\dot{x}$, is compared with $\hat{y} \equiv C\dot{\hat{x}}$, and their difference is used to serve as a correcting term. The difference of $y$ and $C\dot{x}$, $y - C\dot{x}$, is multiplied by an $n \times q$ real constant matrix $L$ and fed into the input of the integrators of the estimator. This estimator will be called as asymptotic state estimator.

The dynamical equation of the asymptotic state estimator shown in figure 5.1 is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\dot{x})$$  \hspace{1cm} (5.2)

which can be written as

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu.$$  \hspace{1cm} (5.3)

Now, define

$$\tilde{x} \equiv x - \hat{x}.$$  \hspace{1cm} (5.4)

Clearly $\tilde{x}$ is the error between the actual state and the estimated state. Subtracting (5.3) from (5.1), we obtain

$$\dot{\tilde{x}} = (A - LC)\tilde{x}.$$  \hspace{1cm} (5.5)

If the eigenvalues of $(A - LC)$ can be chosen arbitrarily, then the behaviour of the error $\tilde{x}$ can be controlled. For example, if all the eigenvalues of $(A - LC)$ have negative real parts smaller than $-\sigma$, then all the elements of $\tilde{x}$ will approaches zero at rates faster than $e^{-\sigma t}$. Consequently, even if there is a large error between $\hat{x}(t_0)$ and $x(t_0)$ at initial time $t_0$, the vector $\tilde{x}$ will approach $x$ rapidly. Thus, if the eigenvalues of $A - LC$ can be chosen properly, an asymptotic state estimator is much more desirable than an open–loop estimator.

5.2.1 Method I

In this method, we apply the design procedures for state feedback to design state estimators. **Theorem 6.** If the $n$–dimensional equation (5.1) is observable, its state can be estimated by using the $n$–dimensional state estimator

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$$

with the error $\tilde{x} = x - \hat{x}$ governed by

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

and all the eigenvalues of $A - LC$ can be arbitrarily assigned, provided complex conjugate eigenvalues appear in pairs.

**Proof.** To show the theorem, it is sufficient to show that if $\{A, C\}$ is observable, then all the eigenvalues of $A - LC$ can be arbitrarily assigned. Indeed, if $\{A, C\}$ is observable, then $\{A', C'\}$, where the prime denotes transposition, is controllable [14]. Hence for any set of $n$ eigenvalues, we can find a real $K$ so that $A' + CK$ has the set as its eigenvalues [14]. Since the eigenvalues of $A + K'C$ and those of $A' + C'K$ are the same, the theorem is established by choosing $L = -K'$.  \hfill $\Box$
5.2.2 Method II

Consider the $n$-dimensional dynamical equation (5.1). It is assumed that it is irreducible. Define the $n$-dimensional dynamical equation

$$\dot{z} = Fz + Gy + Hu$$  \hspace{1cm} (5.6)

where $F$, $G$ and $H$ are respectively, $n \times n$, $n \times p$, and $q \times n$ real constant matrices.

**Theorem 7.** The state $z(t)$ in (5.6) is an estimate of $Tx(t)$ for some $n \times n$ real constant matrix $T$ in the sense that $z(t) - Tx(t) \to 0$ as $t \to \infty$, for any $x(0)$, $z(0)$, and $u(t)$ if and only if

1. $TA - FT = GC$
2. $H = TB$
3. All the eigenvalues of $F$ have negative real parts.

**Proof.** Define

$$e \equiv z - Tx.$$

Then we have

$$\dot{e} = \dot{z} - T\dot{x} = Fz + Gy + Hu - TAx - TBu = Fe + (FT - TA + GC)x + (H - TB)u.$$

If the three conditions in (5.7) are met, then $e(t) = e^{Ft}e(0) \to 0$ for any $x(0)$, $z(0)$, and $u(t)$. Hence $z(t)$ is an estimate of $Tx(t)$.

Now, we show the necessity of the conditions. If 3 is not met, then for $x(0) = 0$ and $u(t) \equiv 0$, we have $e(t) = e^{Ft}z(0) \not\to 0$ as $t \to \infty$. If $H \neq TB$, we can find a $u(t)$ to make $e(t) \not\to 0$ as $t \to \infty$. If $TA - FT \neq GC$ and if $\{A, B\}$ is controllable, we can find a $u(t)$ to generate a $x(t)$ which makes $e(t) \not\to 0$ as $t \to \infty$. This establishes the necessity part of the theorem.

Thus the following algorithm can be proposed.

**Algorithm 8.**

1. Choose an $F$ so that all of its eigenvalues have negative real parts and are disjoint from those of $A$.
2. Choose a $G$ so that $\{F, G\}$ is controllable.
3. Solve the unique $T$ in $TA - FT = GC$.
4. If $T$ is nonsingular, compute $H = TB$. The equation in (5.6) with these $F$, $G$, and $H$ is an estimate of $Tx(t)$ or $\hat{x}(t) = T^{-1}z(t)$. If $T$ is singular, choose different $F$ and/or $G$ and repeat the process.
We give some remarks regarding this algorithm. If $A$ and $F$ have no common eigenvalue, then a solution $T$ always exists in $TA - FT = GC$ and is unique [14]. If $A$ and $F$ have common eigenvalues, then a solution $T$ may or may not exist in $TA - FT = GC$. To remove this uncertainty, we require $F$ and $A$ to have no common eigenvalues.

The controllability of $\{F, G\}$ is needed for the existence of a nonsingular $T$, as will be established in the following theorem.

**Theorem 9.** If $A$ and $F$ have no common eigenvalues, necessary conditions for the existence of a nonsingular $T$ in $TA - FT = GC$ are $\{A, C\}$ observable and $\{F, G\}$ controllable. For the single-output case, this conditions are sufficient as well.

**Proof.** See [14].

Note that the computational problem of $T$ in $TA - FT = GC$ has been studied extensively in the literature and its solution may not be desirable.

The design procedure in Method I requires the transformation of a dynamical equation into a multivariable observable form. This step is by no means simple numerically. It is not clear at present which method, I or II, is better in terms of computational efficiency and numerical stability.

### 5.3 A numerical application

In the previous chapters of this thesis we have dealt with the problem of designing a decentralized constant-volume controller for open channels. In all the numerical simulations carried on, volume variations have always been considered as measurable variables. Obviously, such an assumption is not realistic. In fact, it has been assumed that volume variations are reconstructed by measuring the water level in a sufficiently large number of points all along the canal reaches.

In this chapter we propose an alternative which reveals to be efficient: we design an asymptotic state observer that is based on the linear decoupled model formulated in Chapter 2. In particular, we design a decentralized volume variation observer that provides an estimation of the volume variation in each canal reach on the base of only local measurements, thus not vanishing the advantage of decentralization in the control. Moreover, since we assume that unknown constant withdrawals may occur, we choose observer gains that are not constant, but contain both a proportional and an integral term.

For numerical simulations we considered the same test canal already used in the previous chapters. We found out that the characteristic dynamical matrices are:

\[
A_1 = -2.1927 \times 10^{-4}, \quad B_1 = \begin{bmatrix} 1.9688 & -1.4314 & 2.1156 & 3.0450 \end{bmatrix},
\]
\[
C_1 = \begin{bmatrix}
0.3289 \\
0.4607
\end{bmatrix} \cdot 10^{-4},
\]

\[
D_1 = \begin{bmatrix}
0.2636 & 0.0703 & -0.1039 & 0.4077 \\
-0.1078 & -0.2315 & 0.3421 & -0.1667
\end{bmatrix},
\]

\[
\bar{A}_2 = -2.4738 \times 10^{-4},
\]

\[
B_2 = \begin{bmatrix}
1.4445 & -0.3550 & -5.1702 & 2.1349
\end{bmatrix},
\]

\[
\bar{C}_2 = \begin{bmatrix}
0.2369 \\
0.2536
\end{bmatrix} \cdot 10^{-4},
\]

\[
D_2 = \begin{bmatrix}
0.3380 & 0.0205 & 1.5991 & 0.4971 \\
-0.0698 & -0.1243 & 0.0205 & -0.1131
\end{bmatrix}.
\]

Now, we want to determine appropriate gain vectors \( \bar{G}_i \), \( i = 1, 2 \), such that the error dynamics

\[
\dot{e}_i = (\bar{A}_i - \bar{G}_i \bar{C}_i) e_i, \quad i = 1, 2,
\]

where \( e_i = v_i - \hat{v}_i, \ i = 1, 2 \), reveals to be satisfactory. Note that, it is immediate to observe that stability holds for whatever couple of gain vectors \( \bar{G}_i \), \( i = 1, 2 \), whose entries are positive real numbers. Moreover, by virtue of the dimensions of \( \bar{A}_i, \bar{C}_i \) and \( \bar{G}_i \) the same eigenvalues for the closed–loop matrices \( \bar{A}_i - \bar{G}_i \bar{C}_i, \ i = 1, 2 \), can be assigned by an infinite number of gain vectors \( \bar{G}_i \).

However, as already announced above, we do not consider constant values for the observer gains. In fact, by taking into account the variations on users withdrawals, the error dynamics are described by the following equations:

\[
\dot{e}_i = (\bar{A}_i - \bar{G}_i \bar{C}_i) e_i - q_{Ci}, \quad i = 1, 2,
\]

thus, null steady state errors can be obtained only if \( \bar{G}_i, \ i = 1, 2 \), are chosen with the following structure:

\[
\bar{G}_i(s) = \begin{bmatrix}
g_{i,P_1} + \frac{g_{i,L_1}}{s} \\
g_{i,P_2} + \frac{g_{i,L_2}}{s}
\end{bmatrix}.
\]

Even in this case, it is easy to prove that stability holds when \( g_{i,P_j}, g_{i,L_j} > 0, \ i, j = 1, 2 \).

In our case these numerical values have been determined via a trial and error procedure. In particular, we obtained:

\[
\bar{G}_1(s) = \begin{bmatrix}
0.8 + \frac{3 \cdot 10^{-6}}{s} \\
0.4 + \frac{3 \cdot 10^{-6}}{s}
\end{bmatrix},
\]

\[
\bar{G}_2(s) = \begin{bmatrix}
2.8 + \frac{5 \cdot 10^{-6}}{s} \\
2 + \frac{4 \cdot 10^{-6}}{s}
\end{bmatrix}.
\]

In the following subsections we present the results of three different simulation test cases. Note that, as in the previous chapters, simulations have been carried out on the completely nonlinear model of open–channels developed at Cemagref [56].
5.3.1 Simulation 1

In this subsection we limit to consider an initial error on the state estimate. We assume that no regulator is acting on the net and no variation on users withdrawals occurs.

The results of simulation 1 are reported in figure 5.2 which presents the percentage errors on the state estimate. As it can be seen, a null steady state error is obtained. Note that, in this case quite the same result would have been obtained even if no integral action had been introduced.

5.3.2 Simulation 2

In the second simulation we assume that a proportional decentralized controller is acting on the system. In particular, we consider the proportional decentralized controller designed via eigenstructure assignment. Note that, similar results can be obtained when implementing the other proportional decentralized controllers presented in the previous chapters of this thesis.

Furthermore, we assume that no initial error is acting on the system. On the contrary, we assume step disturbances acting on both canal reaches after a time period equal to 20'. In particular, we consider $q_{c1} = -0.3 \, \text{m/s}^2$ and $q_{c2} = -0.25 \, \text{m/s}^2$.

The results of simulation 2 have been reported in figure 5.3. Figure 5.3.a) presents the percentage volume variations, whose steady state values differ from zero since no integral action has been introduced in the control law. Figure 5.3.b) presents the percentage errors on the state estimate. As it can be seen, steady state errors are equal to zero thanks to the integral action on the observer gains. Finally, in figure 5.3.c) variations on gate openings have been reported.
Figure 5.3: The results of Simulation 2.
5.3.3 Simulation 3

In the second simulation we assume that a PI decentralized controller is acting on the system. In particular, we consider the PI decentralized controller designed via eigenstructure assignment. Note that, similar results can be obtained when implementing all the other proportional integral decentralized controllers presented in the previous chapters of this thesis.

As in the previous simulation test case, we assume that no initial error is acting on the system. On the contrary, we assume step disturbances acting on both canal reaches after a time period equal to 20'. In particular, we consider \( q_{c1} = -0.3 \text{ m/s}^2 \) and \( q_{c2} = -0.25 \text{ m/s}^2 \).

The results of simulation have been reported in figure 5.4. Figure 5.4.a) presents the percentage volume variations, whose steady state is equal to zero thanks to the integral action introduced in the control law. Figure 5.4.b) presents the percentage errors on the state estimate. As it can be seen, steady state errors are equal to zero thanks to the integral action on the observer gains. Finally, in figure 5.4.c) variations on gate openings have been reported.
All the numerical simulations presented above enable us to conclude that satisfactory results can be obtained even if state estimators are considered. However, in such a case transients reveal to be much longer, as well as the maximum variations on state components reveal to be much more relevant.

5.4 Conclusions

In this chapter we dealt with the problem of designing an asymptotic state observer. In particular the aim of the author was that of designing decentralized asymptotic state observers, that is to say, state observers capable of reconstructing the state estimation on the base of only local measurements, thus not vanishing the advantage of decentralization in the control. The introduction of an integral action in the observers gains also enabled us to obtain a null steady state error on the volume estimates even in presence of unknown constant variations on users withdrawals.

The results of some numerical simulations have also been presented to demonstrate the effectiveness of the proposed observers.
Chapter 6

Stability robustness analysis

6.1 Introduction

The main purpose of using feedback is to reduce the effects of uncertainty. Some uncertainty is always present, both in the environment of the system – we do know in advance exactly what disturbance and noise signals the system will be subjected to, and in the behaviour of the system itself – we know our models are not perfect, and the behavior of the system may change in unpredictable ways. A further reason for using feedback is to stabilize an unstable system.

Classical feedback–system design dealt with the problem of plant uncertainty by prescribing stability margins, by means of specified gain or phase margins or specified peak M values. Implicit in the use of such margins was a rather crude model of the uncertainty to which the plant was subject. If the only real concern was to maintain stability, and the prescribed phase margin was 40°, for example, then there was an implication that the plant model may be underestimating phase lag by as much as 40° at the cross–over frequency. Such a description of uncertainty is unstructured in the sense that it bounds the magnitude of possible perturbations, but does not trace the origins of the perturbations to specific elements of the plant.

In the following section we are concerned with making explicit models of unstructured uncertainty. Then, we will discuss the stability robustness.

Finally, we apply all these concepts to the applicative case at hand. In particular, we study whether the proposed feedback designs are robust when output multiplicative perturbations occur, due to the low–frequency approximations and to the variations with respect to the reference configuration of uniform flow.

It is also shown how the introduction of the observer does not affect stability.
6.2 Unstructured uncertainty

For multivariable plant the three most commonly used models are as follows: let $G_0(s)$ be a nominal transfer function, which is the best estimate, in some sense, of the true plant behaviour, and let $G(s)$ denote the true transfer function of the plant; then

\[ G(s) = G_0(s) + \Delta_a(s) \quad (6.1) \]
\[ G(s) = G_0(s)[I + \Delta_i(s)] \quad (6.2) \]

or

\[ G(s) = [I + \Delta_o(s)]G_0(s) \quad (6.3) \]

where $\Delta_a$ represents an additive perturbation, $\Delta_i$ an input multiplicative perturbation, and $\Delta_o$ an output multiplicative perturbation.

The only restriction on the perturbations is on their 'size', which is measured by $\|\Delta\|_\infty$.

The additive model (6.1) may be used to pose some robust stabilization problems which have nice solutions [39], but the multiplicative models (6.2) and (6.3) are often more realistic, since $\|\Delta_i\|_\infty$ and $\|\Delta_o\|_\infty$ represent relative rather than absolute magnitudes. For example, $\|\Delta\|_\infty \leq 0.1$ implies that the size of the perturbation is at most 10% of the 'size' of $G_0$, since

\[ \|G - G_0\|_\infty = \|G_0\Delta_i\|_\infty \leq \|G_0\|_\infty \|\Delta_i\|_\infty \leq 0.1\|G_0\|_\infty. \quad (6.4) \]

However, specifying $\|\Delta_a\|_\infty \leq 0.1$ implies that

\[ \|G - G_0\|_\infty = \|\Delta_a\|_\infty \leq 0.1. \quad (6.5) \]

We need both models (6.2) and (6.3) because multiplication of transfer-function matrices is non-commutative. In effect, (6.2) pretends that all the uncertainty occurs at the plant input, while (6.3) pretends that it all occurs at the output.

In some cases it will not be appropriate to refer all the uncertainty to the input or the output. It is then necessary to use either the additive model (6.1), or some other multiplicative model such as

\[ G(s) = [I + \Delta_o(s)]G_0(s)[I + \Delta_i(s)] \quad (6.6) \]

or

\[ G(s) = G_0(s) \ast [I + \Delta_a(s)] \quad (6.7) \]

where $\ast$ represents the Schur product, namely element-by-element multiplication. Among other models which have been proposed are the inverse multiplicative model

\[ G(s) = [I + \Delta_R(s)]^{-1}G_0(s) \quad (6.8) \]

and a model based on matrix fraction descriptions:

\[ G(s) = N(s)D^{-1}(s) \quad (6.9) \]
where

\[ N(s) = N_0(s) + \Delta_N(s) \quad \text{and} \quad D(s) = D_0(s) + \Delta_D(s). \quad (6.10) \]

A major advantage of using operator norms to describe model uncertainty is that it is not necessary to postulate the existence of a 'true' transfer–function model. Real plants are always non-linear, and frequently time-varying, so no 'true' transfer function description can exist. But we can interpret a model of uncertainty, such as (6.1), in the following way. Suppose that we apply an input \( u(t) \) to a plant with \( \|u\|_2 = 1 \), and that the resulting output is \( y(t) \). Suppose also that the same input applied to a model with transfer function \( G_0(s) \) gives the output \( y_0(t) \). Then (6.1) really says little other than

\[ \|y - y_0\|_2 \leq \|\Delta u\|_{\infty}. \quad (6.11) \]

### 6.3 Stability robustness

In this section we turn our attention to ways of checking whether a particular feedback design is robust in the face of unstructured uncertainty. The fundamental property which must be retained for all possible perturbations of the plant (and the compensator) is stability of the feedback system.

In particular, we limit ourselves to consider the output multiplicative model (6.3), being the only one of interest when dealing with our application.

In the following we shall allow that \( \Delta_o \) be unstable, but impose that the nominal plant \( G_0 \) and the perturbed plant \( G \) have the same number of unstable poles. Since the loop is stable for \( \Delta_o = 0 \), then by assumption, and since \( G \) and \( G_0 \) always have the same number of unstable poles, the loop will remain stable provided the number of encirclements of \( -1 \) by the characteristic loci of \( GK \) remains unchanged, which it will if no locus passes through \( -1 \) as \( G \) varies; in other words

\[ \det[I + G(j\omega)K(j\omega)] \neq 0 \quad (6.12) \]

for any \( \omega \) and any permissible \( G \). Now, (6.12) is the same as

\[ \sigma[I + G(j\omega)K(j\omega)] > 0 \quad (6.13) \]

for all \( \omega \), which is the same as

\[ \sigma[I + G_0K + \Delta_oG_0K] > 0 \quad (6.14) \]

or

\[ \sigma[(\Delta_oG_0K)^{-1} + \Delta_o^{-1} + I|\Delta_oG_0K] > 0 \quad (6.15) \]

which holds if

\[ \sigma[(G_0K)^{-1} + I|\Delta_o^{-1} + I] > 0. \quad (6.16) \]
It can be proved [53] that this will hold if
\[ \sigma\{|(G_0K)^{-1} + I|\Delta_o^{-1}\} > 1 \] (6.17)
which is implied by
\[ \sigma(\Delta_o^{-1})\sigma\{|(G_0K)^{-1} + I\} > 1. \] (6.18)
Hence
\[ \sigma(\Delta_o)\sigma\{|(G_0K)^{-1}\} < 1 \] (6.19)
or
\[ \sigma\{G_0K[I + G_0K]^{-1}\} < \frac{1}{\sigma(\Delta_o)}. \] (6.20)

Note that we have established the sufficiency of (6.20) for robust stability, but it can again be shown to be a necessary condition, if all permissible perturbations may actually occur.

Stability condition (6.20) is a special case of a very general result, known as the **small gain theorem**, which states that a feedback loop composed of stable operators will certainly remain stable if the product of all the operator gains is smaller than unity. This is true for nonlinear as well as linear operators.

### 6.4 Numerical applications

In this section we present two different robustness analysis. In the first subsection we study whether the proposed feedback designs are robust when output–multiplicative perturbations occur due to the low–frequency approximations and to the variations with respect to the reference configuration of uniform flow. In the second subsection we study the same problem when also the state observer is introduced. For numerical simulations we consider the same test canal already used in the previous chapters.

#### 6.4.1 Robustness analysis with no state observer

As it has been shown in [21, 85, 89] it is possible to measure the "distance" between the reference model \((G(j\omega))\) and the approximate nominal one \((G_A(j\omega))\). As usual, the relationship between the two transfer function matrices can be written as follows:
\[ G(j\omega) = |I + L(j\omega)|G_A(j\omega). \] (6.21)

Matrix \(L(j\omega)\) bounds \(G\) in a normalized neighborhood of \(G_A\) and describes the uncertainty in the output-multiplicative form. Here the transfer function matrices \(G\) and \(G_A\) are known, and equation (6.21) can therefore be solved for \(L(j\omega)\). The largest singular value \(\sigma[L(j\omega)]\) is taken as the matrix \(L(j\omega)\) norm.
In this section we are interested in finding out whether a particular feedback design is robust for this output-multiplicative perturbation. In our case \( L(j\omega) \) is derived from the low-frequency approximations and contains all the neglected high-frequency modes.

At this purpose we use the sufficient condition demonstrated in [27] and presented in the previous section. It requires that the relationship:

\[
\frac{1}{\sigma[G_A(j\omega)K(j\omega)(I + G_A(j\omega)K(j\omega))^{-1}] > \sigma[L(j\omega)]}
\]

holds \( \forall \omega \geq 0 \).

This kind of analysis allows us to demonstrate the robustness of the reference closed-loop system with respect to perturbations generated by the series expansion, that is, by all the neglected high-frequency modes.

In this way the perturbations caused by variations with respect to the reference configuration chosen to derive the nominal model \( G_A \), are not taken into account. Therefore, to complete the robustness analysis we should establish a range of reference configuration variations and verify that relationship (6.22) holds for each configuration in the chosen set. This is equivalent to defining \( \sigma[L(j\omega)] \) as follows [85]:

\[
\sigma[L(j\omega)] = \max_k \sigma[G_k(j\omega)G_A^{-1}(j\omega) - I]
\]

as \( k \) varies in the set of reference configurations. In other words, the curve \( \sigma[L(j\omega)] \) represents the envelope of the different \( \sigma_k[L(j\omega)] \)'s, each corresponding to a variation in configuration.

The approach followed here, like the one in [21, 85, 89], based on a set of varying configurations, of course involves considering the corresponding ranges over which the coefficients of the model \( G \) vary. These variations are strongly correlated by the very fact that they refer to possible configurations. This procedure has therefore been preferred to the approach whereby parameter variations are regarded as being completely independent of one another. In the latter case the results obtained are unduly restrictive because one considers many combinations of parameter values that are unrealistic.

Obviously before proceeding with the robustness analysis we have to prove that the nominal plant \( G_A \) and the perturbed plants \( G_k \) have the same number of unstable poles for all \( k \).

Note that stability of the closed-loop reference models could also have been studied by determining the analytical expressions of the closed loop transfer function matrix for each reference configuration. However, since two matrix inversions must be performed, the final expressions would have been very complicated and it would have been difficult to evaluate the presence of unstable poles.
Stability of the nominal model

By evaluating the eigenvalues of matrix $A$ in equation (2.97) we can affirm that the reference model $G_A$ is stable.

To perform robustness analysis it is necessary to prove that stability holds for each $G_k$, as $k$ varies in the set of reference configurations.

Stability of the reference model

The stability analysis of each reference model can be summarized in the following items. Since the steps are the same for all $G_k$’s, we shall omit the subscript $k$.

1. $A_1(s)$ and $A_2(s)$ have no positive poles. In fact there exist no $s$ with $Re(s) > 0$ such that the common denominator of the $A_{ji}$’s is null in subcritical flow (i.e. $c_{0i} > y_{0i}$, which is our case), for $j = 1, \cdots, 4$, $i = 1, 2$. This assumption can be proved by demonstrating that:

$$e^{\alpha_1 l_i} = e^{\alpha_2 l_i}, \quad (6.24)$$

holds only if:

$$Re(\alpha_1 l_i) = Re(\alpha_2 l_i). \quad (6.25)$$

It is easy to observe that (6.25) happens only if the square root in $\alpha_1, \alpha_2$ is a complex number, that is, if the term under the square root is a negative real number $1$. But this can occur only for $s$ with $Re(s) \leq 0$.

2.

$$A(s) = I^* A_3(s)$$

$$A_3(s) = [I - \Delta A_2(s)]^{-1} \Delta A_1(s)$$

As a consequence of Step 1 the presence of positive poles in $A_3(s)$ can only be due to $[I - \Delta A_2(s)]^{-1}$.

Let us consider the transfer function matrix:

$$W_3(s) = [I - \Delta A_2(s)]^{-1} \Delta.$$  

It can be regarded as the closed loop transfer function matrix of the system in figure 6.1. Based on the Nyquist criterion for multivariable systems [53] it can be proved that $W_3(s)$ has a positive pole introduced by $1/d_3(s)$, where

$$d_3(s) = det \ (I - \Delta A_2(s)) \cdot$$

$1\alpha_{1,2}$ can be written as $\alpha_{1,2} = a + jb \pm (c + jd)$ where $a$, $b$, $c$, $d \in \mathbb{R}$. Therefore (6.25) holds only if $a + c = a - c$, i.e., $c = 0$. 

108
Figure 6.1: Scheme of closed–loop system $W_3(s)$.

3. Let us consider the following transfer function matrix:

$$W_1(s) = [sI - A(s)]^{-1} = \left[ I - \frac{A(s)}{s} \right]^{-1} \frac{I}{s}.$$ 

It can be thought of as the closed loop transfer function matrix of the system in figure 6.2. Based on the Nyquist criterion it can be proved that $W_1(s)$ is stable while the open loop transfer function matrix $A(s)/s$ has a positive pole.

Let us define:

$$d_1(s) = \det \left( I - \frac{A(s)}{s} \right).$$

$d_1(s)$ has the following structure:

$$d_1(s) = \frac{1}{d_3(s)} f(s)$$

where $f(s)$ is a function with no positive zeros. Each element in $\text{adj}(I - A(s)/s)$ contains $1/d_3(s)$ as a factor. If $W_1(s)$ is stable it can be concluded that $1/d_3(s)$ is simplified with $1/d_1(s) = d_3(s)^2/f(s)$ and

$$W_1(s) = d_3(s) M(s)$$

where $M(s)$ is a stable transfer function matrix.

4.

$$B(s) = \Gamma^* B_3(s)$$

$$B_3(s) = [I - \Delta A_2(s)]^{-1} \Gamma$$

It follows from Step 2 that $B(s)$ has a positive pole introduced by $1/d_3(s)$, thus

$$B_3(s) = \frac{1}{d_3(s)} N(s)$$

where $N(s)$ is stable.
Figure 6.2: Scheme of closed-loop system $W_1(s)$.

5.

$$G(s) = [sI - A(s)]^{-1}B(s) = W_1(s)B(s) = M(s)N(s)$$

Therefore we can conclude that the reference model too is stable as it is the product of two stable transfer function matrices.

Numerical results

Now, let us discuss the closed loop system robustness when applying the decentralized control laws obtained via $H_2$ norm minimization [85, 89]. Note that similar results can be obtained when considering all the other decentralized controllers proposed in the previous chapters. Table 6.3 shows the values of the variables of interest in the nominal configuration (underlined) and in the perturbed configurations for a change in uniform flow water level depth of $\pm 10\%$ with respect to the nominal configuration. A total of nine configurations were examined (including the nominal one), and they were all compatible ($q_{ci} \geq 0, i = 1, 2$). The flow rates were calculated using the Chezy formula, taking the roughness coefficient to be 0.36.

In figure 6.4 the $\bar{\sigma}[L(j\omega)]$ curve is plotted (thin line), which represents the envelope of the different $\bar{\sigma}_k[L(j\omega)]$'s for the nine configurations examined. In the specific case, the errors are quite modest, and their mean value for $\omega > 2 \cdot 10^{-2}$ remains practically constant even for frequencies higher than those reported in the figure. The other curve (thick line) represents the first member of equation (6.22) both in the case of PI decentralized control law. By looking at figure 6.4 we can conclude that the reference closed loop system is stable even in the presence of the considered variations with respect to the reference configuration.

6.4.2 Robustness analysis with asymptotic state observer

In this subsection we want to discuss the closed loop system robustness when applying both the controller and the observer.
Figure 6.3: Values of the actual variables of interest in the nominal (underlined) and in the perturbed configurations.

<table>
<thead>
<tr>
<th>$h_{01}$ (m)</th>
<th>$h_{02}$ (m)</th>
<th>$q_{01}$ (m$^3$/s)</th>
<th>$q_{02}$ (m$^3$/s)</th>
<th>$q_{0\gamma 1}$ (m$^3$/s)</th>
<th>$q_{0\gamma 2}$ (m$^3$/s)</th>
<th>$\lambda_{01}$ (m$^3$/s)</th>
<th>$\lambda_{02}$ (m$^3$/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.53</td>
<td>1.20</td>
<td>2.47</td>
<td>2.35</td>
<td>0.66</td>
<td>1.84</td>
<td>1.98</td>
<td>2.97</td>
</tr>
<tr>
<td>1.32</td>
<td>1.08</td>
<td>3.62</td>
<td>1.20</td>
<td>0</td>
<td>2.50</td>
<td>1.61</td>
<td>1.18</td>
</tr>
<tr>
<td>1.70</td>
<td>1.20</td>
<td>5.94</td>
<td>3.02</td>
<td>2.92</td>
<td>0.15</td>
<td>2.50</td>
<td>1.61</td>
</tr>
<tr>
<td>1.32</td>
<td>1.08</td>
<td>3.62</td>
<td>2.32</td>
<td>0</td>
<td>2.50</td>
<td>1.61</td>
<td>2.21</td>
</tr>
<tr>
<td>1.87</td>
<td>1.20</td>
<td>7.19</td>
<td>3.02</td>
<td>4.17</td>
<td>0.15</td>
<td>3.41</td>
<td>1.39</td>
</tr>
<tr>
<td>1.32</td>
<td>1.08</td>
<td>3.62</td>
<td>3.57</td>
<td>0</td>
<td>3.41</td>
<td>1.39</td>
<td>1.84</td>
</tr>
</tbody>
</table>

Figure 6.4: Envelope of the different $\bar{\sigma}_k[L(j\omega)]$’s for the configurations examined (thin line); $1/\bar{\sigma}[G_A(j\omega)K(j\omega)[I + G_A(j\omega)K(j\omega)]^{-1}]$ in the case of PI decentralized control (thick line).

111
To take into account also the presence of the state observer that, as is well known, can affect stability, we have to represent the whole closed loop system as in figure 6.5 where

\[
\begin{align*}
\mathbf{u} &= \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix}, \\
\bar{\mathbf{y}} &= \begin{bmatrix} h_{A1} \\ h_{B1} \\ \vdots \\ h_{An} \\ h_{Bn} \end{bmatrix}, \\
\hat{\mathbf{x}} &= \begin{bmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_n \end{bmatrix},
\end{align*}
\]

being \( \mathbf{u} \) the input vector, \( \bar{\mathbf{y}} \) a vector whose entries represent the upstream and downstream variations in each pool, and \( \hat{\mathbf{x}} \) the vector of state estimations, used for the feedback law.

Now, let us first consider the reference model. In such a case \( F_1(s), F_2(s) \) and \( \hat{F}(s) \) are transcendental transfer function matrices and can be easily determined as a function of the system dynamical matrices introduced in chapter 2. In fact,

\[
\begin{align*}
V(s) &= G(s)\Sigma(s) = I^*Q(s) \\
Q(s) &= I\Sigma(s) + \Delta H(s) \\
G(s)\Sigma(s) &= I^*\Gamma\Sigma(s) + I^*\Delta H(s) \\
(G(s) - I^*\Gamma)\Sigma(s) &= I^*\Delta H(s) \\
H(s) &= (I^*\Delta)^{-1}(G(s) - I^*\Gamma)\Sigma(s)
\end{align*}
\]

thus

\[
F_1(s) = (I^*\Delta)^{-1}(G(s) - I^*\Gamma).
\]

In the previous section it has been proved that \( G(s) \) has no unstable poles. Therefore, we can conclude that the same holds for \( F_1(s) \).

Now, let us determine matrix \( \hat{F}(s) = F_1(s)F_2(s) \). The observer dynamics is regulated by the following state equation:

\[
\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\bar{\mathbf{u}}(t) + \bar{G}(s)(\bar{\mathbf{y}}(t) - \bar{\mathbf{y}}(t))
\]

Figure 6.5: Block–diagram of the closed–loop system with observer.
where
\begin{equation}
\dot{y}(t) = \dot{C}x(t) + \dot{D}u(t) \tag{6.33}
\end{equation}
and \(\tilde{G}(s)\) is the observer matrix. Thus,
\begin{equation}
\dot{x}(t) = \tilde{A}\dot{x}(t) + \tilde{B}\dot{u}(t) + \tilde{G}(s)\dot{y}(t) - \tilde{G}(s)(\dot{C}\dot{x}(t) + \dot{D}u(t)) \tag{6.34}
\end{equation}
and inverse L–transforming the above equation, we get
\begin{equation}
(sI - \tilde{A} + \tilde{G}(s)\tilde{C})\tilde{X}(s) = \tilde{G}\tilde{Y}(s) + (\tilde{B} - \tilde{G}(s)\tilde{D})\tilde{U}(s) \tag{6.35}
\end{equation}
where \(\tilde{X}(s), \tilde{Y}(s)\) and \(\tilde{U}(s)\) are the Laplace transforms of \(\dot{x}(t), \dot{y}(t)\) and \(\dot{u}(t)\), respectively. Moreover, \(\tilde{U}(s)\) can be written as a linear combination of \(\Sigma(s)\) and \(\tilde{Y}(s)\), i.e.,
\begin{equation}
\tilde{U}(s) = I_1\Sigma(s) + I_2\tilde{Y}(s), \tag{6.36}
\end{equation}
where \(I_1\) and \(I_2\) are appropriate matrices whose entries are all 1’s and 0’s. Equation (6.35) can be rewritten as
\begin{equation}
(sI - \tilde{A} + \tilde{G}(s)\tilde{C})\tilde{X}(s) = \tilde{G}\tilde{Y}(s) + (\tilde{B} - \tilde{G}(s)\tilde{D})I_1\Sigma(s) + (\tilde{B} - \tilde{G}(s)\tilde{D})I_2\tilde{Y}(s). \tag{6.37}
\end{equation}
Finally, being \(\tilde{Y}(s) = \tilde{F}_1(s)\Sigma(s)\),
\begin{equation}
\tilde{F}(s) = (sI - \tilde{A} + \tilde{G}(s)\tilde{C})^{-1}\{\tilde{G}(s)\tilde{F}_1(s) + (\tilde{B} - \tilde{G}(s)\tilde{D})I_1 + (\tilde{B} - \tilde{G}(s)\tilde{D})I_2\tilde{F}_1(s)\}. \tag{6.38}
\end{equation}

At this point we can conclude that matrix \(\tilde{F}(s)\) is stable. In fact, two different cases may occur. If \(\tilde{G}\) is constant, then \((sI - \tilde{A} + \tilde{G}\tilde{C})^{-1}\) is a diagonal matrix whose poles have negative real parts; the poles of matrix between parenthesis \{\cdots\} are the same as those of \(\tilde{F}_1(s)\), thus they are all stable.

In the case of PI decentralized observer, quite the same reasoning can be repeated with only an additional consideration: the poles in the origin that are present in \{\cdots\} are simplified with the zeros in \((sI - \tilde{A} + \tilde{G}\tilde{C})^{-1}\). Thus, even in this case \(\tilde{F}(s)\) is stable.

Note that exactly the same reasoning can be repeated for all \(\tilde{G}_k(s)\)'s, as \(k\) varies in the set of reference configurations.

If we consider the nominal approximate model, we can compute a rational transfer function matrix \(\tilde{F}_A(s)\), that is the counterpart of the transcendental transfer function matrix \(\tilde{F}(s)\). Obviously, the stability of \(\tilde{F}_A(s)\) can be proved by virtue of the same considerations.

By replacing \(\tilde{G}(s)\) and \(\tilde{G}_A(s)\) with \(\tilde{F}(s)\) and \(\tilde{F}_A(s)\), respectively, we can repeat the same robustness analysis as in the previous section. In fact, even in this case it is possible to measure the "distance" between \(\tilde{F}(j\omega)\) and \(\tilde{F}_A(j\omega)\) and we can write:
\begin{equation}
\tilde{F}(j\omega) = [I + L_u(j\omega)]\tilde{F}_A(j\omega). \tag{6.39}
\end{equation}
Matrix $L_o(j\omega)$ bounds $\hat{F}$ in a neighborhood of $\hat{F}_A$ and describes the uncertainty in the output–multiplicative form. Here the transfer function matrices $\hat{F}$ and $\hat{F}_A$ are known, thus equation (6.39) can be solved for $L_o(j\omega)$ and we have to verify that the relationship:

$$\frac{1}{\sigma[\hat{F}_A(j\omega)K(j\omega)[I + \hat{F}_A(j\omega)K(j\omega)]^{-1}]} > \sigma[L_o(j\omega)]$$

holds $\forall \omega \geq 0$.

As in the previous section, robustness analysis is performed both with respect to the high frequency modes and with respect to the variations of nominal configuration reported in table 6.3. As already discussed above, this is equivalent to defining

$$\sigma[L_o(j\omega)] = \max_k \sigma[\hat{F}_k(j\omega)\hat{F}_A^{-1}(j\omega) - I]$$

as $k$ varies in the set of reference configurations. In other words, the curve $\sigma[L_o(j\omega)]$ represents the envelope of the different $\sigma_k[L_o(j\omega)]$’s, each corresponding to a variation in configuration.

In figure 6.6 the $\sigma[L_o(j\omega)]$ curve is plotted (thin line), which represents the envelope of the different $\sigma_k[L_o(j\omega)]$’s for the nine configurations examined. In the specific case, the errors are quite modest, and their mean value for $\omega > 8 \cdot 10^{-3}$ remains practically constant even for frequencies higher than those reported in the figure. The other curve (thick line) represents
the first member of equation (6.40) in the case of the PI decentralized control law designed via $H_2$ norm minimization [85, 89] and the PI observer. By looking at figure 6.6 we can conclude that the reference closed loop system is stable even in the presence of the considered variations with respect to the reference configuration. Note that similar results can be obtained when considering the other control laws and the simpler proportional observer.

An important remark needs to be done. The above robustness analysis enabled us to demonstrate the boundness of the estimated state vector $\hat{x}$. Moreover, the stability of $F_1$ and $\hat{F}$, and thus that of $F_2$, enables us to conclude that also $\hat{y}$ is bounded. This obviously implies the boundness of $x = v$, as we want to prove.

### 6.5 Conclusions

In this chapter we dealt with the problem of stability robustness in presence of unstructured uncertainty on the plant model. Perturbations have been distinguished in additive, input multiplicative and output multiplicative, depending on the relationships between the true and the nominal transfer function matrices. Then, in the case of output multiplicative model of the perturbations, a sufficient condition for robust stability can be derived and the assumption about the stability of the perturbation is relaxed. However, in such a case it is necessary to assume that the nominal and the perturbed plant have the same number of unstable poles.

The above results have been applied to the applicative case of interest in this thesis. In particular, we have shown that the proposed feedback designs are robust when output multiplicative perturbations occur, due to the low-frequency approximations and to the variations with respect to the reference configuration of uniform flow.

A further detailed robustness analysis also enabled us to conclude that stability is not affected by the introduction of the observer, neither the proportional, nor the proportional integral one.
Conclusions

In this thesis the problem of designing both a proportional and a proportional integral decentralized constant volume controller for open–channels has been accurately examined. The whole work can be divided in the following four main parts.

• Firstly, a careful examination of all the existing models of the physical system at hand has been carried out. It has been evaluated that two models deduced from the Saint–Venant equations by Corriga et al. in [18, 19, 21] provide a valid solution to the modeling problem. The first one, denoted as the reference model, expresses the dynamic relationships, in terms of transcendental functions, between the gate opening sections and the corresponding stored volume variations in the different canal reaches with respect to an initial reference configuration of uniform flow. The second one, denoted as approximate nominal model, is obtained from the previous one by means of a series expansion around $s = 0$, where the most significant phenomena take place. It is a state variable linear and time invariant model whose state variables and control variables are equal to the output and input variables of the previous model, respectively.

• Secondly, the problem of designing a decentralized controller has been widely investigated. Different design techniques have been examined [84, 85, 86, 88, 89, 90, 93, 94] and implemented on an applicative example. Note that all numerical simulations have been carried out on the commercial SIC software [56], a completely nonlinear model of open–channels developed at Cemagref (Montpellier, France). All of them revealed to be particularly efficient when applied to the case of interest and it is not easy to establish in general which is the most efficient synthesis procedure. Three of these design techniques [84, 85, 86, 88, 89, 90] solve the synthesis problem as a problem of parameter optimization, while in the last one the controller is determined via eigenstructure assignment [93, 94]. We can observe that from a theoretical point of view the latter one, as well as the procedure based on the $H_2$ norm minimization, is surely much more sophisticated with respect to the others. On the contrary its implementation is computationally less convenient and can also become inaccurate when applied to great order systems.

• Thirdly, the problem of designing an asymptotic state observer has been examined [92]. For this purpose, a new linear decoupled dynamical model of open–channels has been
formulated. It enabled us to design a linear observer for each canal reach, thus making it possible to reconstruct the state variables on the base of only local measurements, thus keeping the advantage of decentralization in the control. Satisfactory results have been obtained both in the presence of initial errors on the volume variation estimates and in the presence of completely unknown users withdrawals.

- Finally, a detailed robustness analysis has been carried out [85, 89]. Both perturbations due to the low frequency approximation and to the variations with respect to the reference configuration of uniform flow have been taken into account. Note that stability has been proved also in the presence of the state observer.

Therefore, the main contribution of this thesis is that of providing efficient PI decentralized controllers that are derived from a model which takes into account the interactions between canal reaches. Moreover, the formulation of a new linear decoupled model of open–channels enables us to design an asymptotic state observer which makes the above controllers interesting for real applications.

Note that two of the synthesis procedures have been firstly proposed by the author in [84, 85, 88, 89] and, as far as the author knows, they have never been presented in the literature before. They can also be interesting when dealing with different applications. Moreover, they can be easily extended to the design of differently structured controllers, as well as to discrete–time systems.
Bibliography


120


[89] C. Seatzu, “Robust PI decentralized control law for open–channel hydraulic systems,”
 Proc. 5th European Control Conference, Karlsruhe (Germany), 1999.


[92] C. Seatzu, G. Usai, “A linear decoupled model of open–channels for the synthesis of
 a decentralized volume variation observer,” 3rd MathMod, Vienna (Austria), 2000 (to
 appear).

[93] C. Seatzu, “Decentralized control of irrigation open–channels via eigenstructure assign-

 eigenstructure assignment,” submitted to Applied Mathematical Modelling.


[96] B.L. Stringam, G.P. Merkley, Field Application of a fuzzy controller for an irrigation canal
 in Roosevelt Utah, Proc. Int. Workshop on the Regulation of Irrigation Canals, Marrakech,

[97] B.L. Stringam, G.P. Merkley, Fuzzy controller simulation for local downstream water level
 control in canals, Proc. Int. Workshop on the Regulation of Irrigation Canals, Marrakech,

[98] W.H. Swann, Constrained optimization by direct search, In Numerical methods for con-
 strained optimization, (Gill P.E. and Murray W., eds), pp. 191–218, London: Academic
 Press.

[99] B. Tomicic, A general operation module for real–time control of surface water resources,
 Master of Science, The International Institute for Hydraulic Engineering, Delft University,
 1989.

 Int. Conf. on Artificial Neural Nets and Genetic Algorithms , Ales, France, 1995.

[101] A. Toudelft, Combining neural and adaptive controllers for a non–minimum phase vary-
 ing time delay system, Journées Hispano–Francaises: Systèmes intelligentes and contrôle
 avancé, University Polytechnique of Catalogne, 1996.

[102] M. Tarokh, Approach to pole assignment by centralized and decentralized output feed-


